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Chapter 1

SPACE TIME CODING IN MULTIPLE INPUT MULTIPLE OUTPUT SYSTEMS: CHALLENGES AND APPLICATIONS

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In a point-to-point communication, the use of multiple transmitter and receiver antennas enables an increased data throughput through spatial multiplexing and an increased range by exploiting the spatial diversity. The design of space time coding schemes that fully achieve the available diversity and the multiplexing gain in a MIMO system has been extensively addressed in literature yielding to the design of the optimal family of codes called perfect space time codes constructed from cyclic division algebra. These codes, originally designed for flat fading channels, received a lot of attention in industry in the last few years. However, the recent standards that use multiple antenna terminals are based on more realistic assumptions involving the use of outer codes and multi-taps channels. This chapter will give a literature overview on the design criteria of space time coding technique and their application in industrial systems.

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1. Introduction

One of the main challenge of the next generations of wireless communication systems is to offer with a high reliability a high data rate. The introduction of multiple antennas at the transmitter and the receiver side, commonly known as multiple input multiple output (MIMO) systems, offers a high data rate through spatial multiplexing and an increased reliability by exploiting the spatial diversity.

These MIMO systems have been widely studied in literature over the last few years aiming to conceive convenient transmission schemes that take advantages from the MIMO benefits. These works consider mainly the cases when no outer codes (*e.g.* convolutional code, turbo code, LDPC, ...) are used at the transmitter side. The design of such schemes depend critically on the availability of the channel state information (CSI) at the transmitter side. When full CSI is available at the transmitter, optimal power allocation can be performed on the different transmit antennas in order to maximize the capacity of the MMO system [1]. However, the full CSIT is not always feasible as it requires a large amount of feedback. The no CSIT assumption is more considered in a practical scenario. For this case, two approaches have been studied in the literature. The first approach proposed by Tarokh *et al.* in [2] is more tailored to the Rayleigh fading distribution and consists to minimize the error probability over all the fading distribution. The second approach proposed by Zheng and Tse in [3] is more general and characterizes at high signal-to-noise ratio (SNR) the dual benefits in term of diversity and spatial multiplexing using the diversity multiplexing tradeoff (DMT) framework.

More recently, Oggier *et al.* in [4] proposed a family of optimal space time codes known as perfect space time codes that fulfill the design criteria of Tarokh *et al.* in [2]. Moreover, it has been shown that these codes are the optimal codes over flat fading MIMO channel (when no outer codes are used) since they achieve full rate and full diversity, preserve the mutual information, achieve the Diversity Multiplexing Tradeoff (DMT) [3] and have a non vanishing determinant [5,6].

Unlike the simplified flat fading MIMO channel without outer codes, industrial transmission schemes are based on more realistic assumptions involving the use of outer codes such as the convolutional code and frequency selective channels [7–9]. The error performances of the perfect space time code in such scenario have been studied in [10] where the authors studied the impact of concatenating space time codes with good outer code in the case of bit interleaved coded modulation and multiple input multiple output system (BICM-MIMO) over a flat or a frequency selective channel.

This chapter will give first a literature overview on the design of transmission schemes for the case of MIMO flat fading channel when no outer code is used. Then the construction of the family of perfect space time codes from the cyclic division algebra is presented. Finally, based on our contribution in [10], we focus on the application of these MIMO codes in a standard context. Throughout this chapter and for the sake of the clarity of the presentation, we will skip all the details related to the mathematical derivations. However, we emphasize on the resulting interpretations and their impact on the MIMO transmission scheme design. The interested reader is invited to refer the indicated references for more details. The rest of the chapter will be organized as following. In Section 2., we review from literature some basic principles on MIMO systems that will be essential for the development

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of this chapter. Section 3. is dedicated to the code construction schemes over the MIMO channel and the evaluation of their error performance over a flat fading MIMO channel. Then, we present in Section 4. the performance of the optimal perfect space time codes in a practical scenario. Finally, we give in Section 5. some concluding remarks and perspectives.

Notation: The notation used in this chapter is as follows. Boldface lower case letters $\mathbf{v}$ denote vectors, boldface capital letters $\mathbf{M}$ denote matrices. $\mathbf{M}^\dagger$ and $\mathbf{M}^T$ denotes respectively the matrix transposition and conjugated transposition operations. $\|\mathbf{v}\|$ stands for the Euclidean norm of vector $\mathbf{v}$. The Frobenius norm of matrix $\mathbf{H}$ is denoted by $\|\mathbf{H}\|_\mathrm{F}^2 = \mathrm{Tr}\{\mathbf{H}^\dagger\mathbf{H}\}$ where $\mathrm{Tr}\{\mathbf{A}\}$ refers to the trace of matrix $\mathbf{A}$. The $N \times N$ identity matrix is represented by $\mathbf{I}_N$. $\mathcal{CN}$ represents the complex Gaussian random variable. Finally, $\mathbb{E}_X\{.\}$ is the mathematical expectation of random variable X.

2. Basic principles on MIMO flat fading channel

The multiple input multiple output (MIMO) systems consist simply to transmit and to receive data from/to different locations over different uncorrelated fading paths. One of the main advantage of the MIMO system is the possibility to recover data over the independent uncorrelated fading paths if one or more paths are deeply faded. This gain is known as diversity gain in the MIMO terminology. Another advantage of the MIMO channel is the gain in data rate due to the spatial multiplexing of data over the different antennas. This gain corresponds to the multiplexing gain in the MIMO terminology.

The main challenge in conceiving a MIMO system is to code data over the different antennas in a convenient way to benefit from MIMO gains.

2.1. From SISO to MIMO channel

2.1.1. Modeling the SISO channel

For a single input single output SISO system, represented in Figure 1, the transmitted signal x is modified by the channel in an unpredicted way due to the multiplicative random fading h and the additive Gaussian noise z with variance N_0.

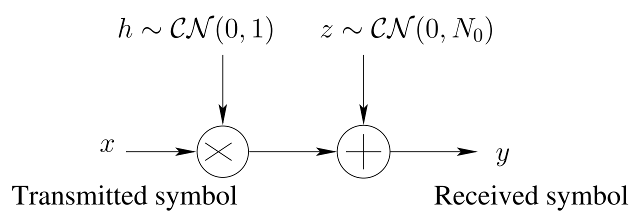

Figure 1: SISO channel

The received signal is therefore

$$y = hx + z.$$

In wireless communication, the fading coefficient is often modeled as a complex Gaussian random variable $h = re^{j\theta} \sim \mathcal{CN}(0,1)$. In this case, r follows a Rayleigh distribution with

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$p(r) = \frac{1}{2\pi} e^{-\frac{r^2}{2}}$ and θ is uniformly distributed in $[0\ 2\pi]$.

2.1.2. Wireless MIMO system

The MIMO system is illustrated in Figure 2. In this case, each antenna j with $j = 1 \ldots n_r$ receives the noisy linear combination of all the transmitted signals. The received signal is,

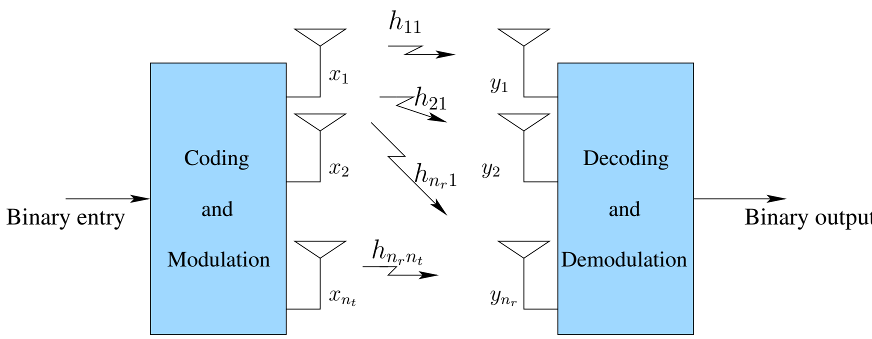

Figure 2: Multiple Input Multiple Output system

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n_r} \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & \ldots & h_{1n_t} \\ h_{21} & h_{22} & \ldots & h_{2n_t} \\ \vdots & & & \\ h_{n_r1} & h_{n_r2} & \ldots & h_{n_rn_t} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n_t} \end{bmatrix} + \begin{bmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_{n_r} \end{bmatrix}$$

Or equivalently, using the matricial notation is,

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \boldsymbol{\omega},$$

where $\mathbf{x} = (x_i)_{1 \le i \le n_t}$ is the transmitted signal subjected to the power constraint $\mathrm{Tr}\left[\mathbb{E}[\mathbf{x}\mathbf{x}^\dagger]\right] \le P$, the channel matrix $\mathbf{H} = (h_{i,j})_{1 \le i \le n_r, 1 \le j \le n_t}$ and $\boldsymbol{\omega}$ is the additive noise vector. We assume here that the channel is flat fading which means that the channel remains constant during all the duration of the transmission.

2.2. Channel capacity and multiplexing gain

The capacity of a channel $C(\mathrm{SNR})$ represents the maximal number of bits that can be transmitted in one time slot where $\mathrm{SNR} = \frac{P}{N_0}$ is the signal to noise ratio. For a MIMO channel, the capacity was derived in [1] and is summarized in Theorem 1.

Theorem 1 (Instantaneous channel capacity)**.** *The maximal rate that can be transmitted over a $n_t \times n_r$ MIMO channel, known as the capacity of MIMO channel is equal to,*

$$C(\mathrm{SNR}) = \max_{\mathbf{Q}:\mathrm{Tr}(\mathbf{Q}) \le P} \log_2 \det\left(\mathbf{I} + \frac{1}{N_0}\mathbf{H}\mathbf{Q}\mathbf{H}^\dagger\right),$$

where $\mathbf{Q} = \mathbb{E}\left[\mathbf{x}\mathbf{x}^\dagger\right]$ is the covariance matrix of the transmitted signal with limited power P such that $\mathrm{Tr}(\mathbf{Q}) \le P$.

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The covariance matrix of the transmitted signal depends on the availability of the channel knowledge $\mathbf{H}$ at the transmitter side.

Definition 1 (Multiplexing gain)**.** *The multiplexing gain is defined as,*

$$r = \lim_{\mathrm{SNR}\to\infty} \frac{\mathbb{E}[C(\mathrm{SNR})]}{\log_2 \mathrm{SNR}} = \min(n_t, n_r),$$

where $\mathbb{E}[C(\mathrm{SNR})]$ *represents the ergodic capacity obtained by averaging over the fading distribution. The multiplexing gain indicates the maximal number of symbols that can be simultaneously transmitted over a channel.*

2.3. Shannon theorem and implications

Theorem 2 (Shannon theorem)**.** *The error-free transmission is possible as long as the transmitted rate R does not exceed the channel's capacity C.*

The main implication of the Shannon theorem is that one can predict the error if the channel's capacity is known at the transmitter side. The knowledge of this capacity at the transmitter side depends on the availability of the channel state information (CSI) at the transmitter side. This CSI is feasible at the receiver side as the receiver can perfectly estimate the wireless channel gains using pilots sequences. However, the situation becomes more complicated at the transmitter side as this CSI is not always feasible at the transmitter side unless a large amount of feedback from the receiver side to the transmitter is considered. The transmission strategy depends hence critically on this CSIT availability.

2.3.1. Full CSIT case: rate adaptation

When full CSIT is available at the transmitter side, the transmitted rate can be adapted to the maximal capacity.

For a given channel realization, the maximal capacity of the MIMO channel can be achieved using the so-called water-filling strategy [1]. This strategy uses the available CSI to jointly diagonalize as shown in Figure 3 at the transmitter and the receiver side the channel $\mathbf{H} = \mathbf{U}\mathbf{\Lambda}\mathbf{V}^\dagger$ by creating as many parallel channels as $n_{\min} = \min(n_t, n_r)$. The transmitted data vector $\tilde{\mathbf{x}}$ that contains $n_{\min}$ symbols is projected on the right eigenvector space $\mathbf{V}$ *i.e.*,

$$\mathbf{x} = \mathbf{V}\tilde{\mathbf{x}},$$

and is subjected to the following power constraint $\mathrm{Tr}\left[\mathbb{E}[\mathbf{x}\mathbf{x}^\dagger]\right] \leq P$. The received vector is projected on the left eigenvector space $\mathbf{U}$, *i.e.*,

$$\begin{aligned}\tilde{\mathbf{y}} = \mathbf{U}^\dagger \mathbf{y} &= \mathbf{U}^\dagger(\mathbf{H}\mathbf{x} + \omega) = \mathbf{U}^\dagger\mathbf{U}\mathbf{\Lambda}\mathbf{V}^\dagger\mathbf{V}\,\tilde{\mathbf{x}} + \mathbf{U}^\dagger\omega,\\ &= \mathbf{\Lambda}\tilde{\mathbf{x}} + \mathbf{z},\end{aligned}$$

where $\mathbf{z} = \mathbf{U}^\dagger\omega \sim \mathcal{CN}(0, N_0\,\mathbf{I})$ is a Gaussian complex vector as $\mathbf{U}^\dagger$ is unitary. Let $\mathbf{P}_x$ denote the diagonal covariance matrix $\mathbf{P}_x = \mathbb{E}[\tilde{\mathbf{x}}\tilde{\mathbf{x}}^\dagger]$. The capacity of this equivalent MIMO channel is then

$$C = \max_{P^*:\sum_i P_i^* \leq P} \sum_{i=1}^{n_{min}} \log\left(1 + \frac{P_i^*\lambda_i^2}{N_0}\right).$$

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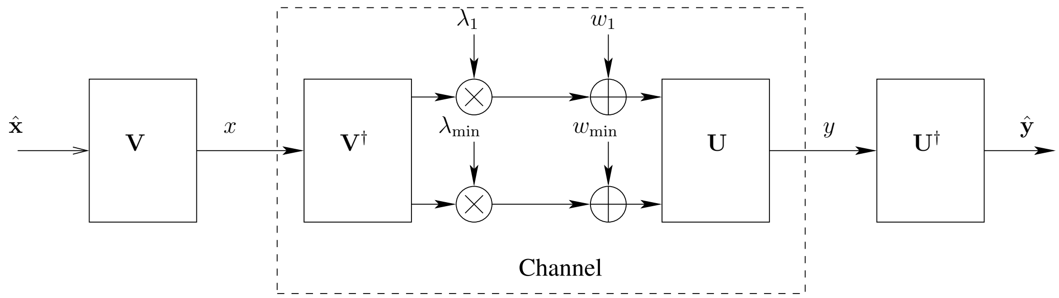

Figure 3: Water-filling: precoder and post-processing

The solution of this convex optimization problem is,

$$P_i^* = \max\left(0, \mu - \frac{N_0}{\lambda_i^2}\right), \tag{1}$$

and the value of μ is determined using the power constraint $\sum P_i^* = P$. The above steps are summarized in Algorithm 1.

Algorithm 1 Water-filling algorithm

1: Perform SVD for the channel $\mathbf{H} = \mathbf{U}\boldsymbol{\Lambda}\mathbf{V}^\dagger$.
2: Compute the power allocated over parallel channel: $\mathbf{P}_x = \mathbb{E}[\tilde{\mathbf{x}}\tilde{\mathbf{x}}^\dagger] = \mu\mathbf{I} - N_0\boldsymbol{\Lambda}^{-2}$.
3: Calculate optimal water level μ : $\mathrm{Tr}\,[\mathbf{P}_x] = P$.
4: Compute the covariance matrix: $\mathbf{K}_x = \mathbb{E}[\mathbf{x}\mathbf{x}^\dagger] = \mathbf{V}\mathbf{P}_x\mathbf{V}^\dagger$.

2.3.2. No CSIT case: diversity techniques

Unlike the CSIT case, optimal power allocation across the antennas cannot be performed in the absence of channel knowledge. In this case, a blind uniform power allocation is used instead, *i.e.*,

$$P_i = \frac{P}{n_t}. \tag{2}$$

The instantaneous channel capacity is then

$$C = \log\det\left(\mathbf{I} + \frac{\mathrm{SNR}}{n_t}\mathbf{H}\mathbf{H}^\dagger\right) = \sum_{i=1}^{m} \log_2\left(1 + \frac{\mathrm{SNR}}{n_t}\lambda_i\right) \tag{3}$$

where λ_i are the eigenvalues of the Wishart channel matrix $\mathbf{H}\mathbf{H}^\dagger$ and $q = \min(n_t, n_r)$. It can be easily observed from (3) that the uniform power allocation does not penalize the maximal multiplexing gain of $\min(n_t, n_r)$ that can be obtained. Using a convenient scheme it is always possible to transmit $\min(n_t, n_r)$ symbols.

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Moreover, the transmitter cannot predict whether the channel is in deep fading or not if no CSI is available at the transmitter side. In this case, there is a non-zero probability that the transmitted rate exceeds the channel capacity and the two events defined in Definitions 2 and 3 should be considered.

Definition 2 (Outage event)**.** *The outage event occurs when the MIMO channel cannot support the transmitted rate R, which means that R exceeds the capacity of the MIMO channel C, i.e.,*

$$O = \left\{ \mathbf{H} : \log\det(\mathbf{I} + \frac{\text{SNR}}{n_t}\mathbf{H}\mathbf{H}^{\dagger}) < R \right\}$$

The outage probability is defined as,

$$P_{out}(R) = P(O)$$

Definition 3 (Error event)**.** *The MIMO system is in error, when the decoded message is different than the transmitted message. The error event is therefore defined as,*

$$\varepsilon = \{\hat{\mathbf{X}} \neq \mathbf{X}\}$$

and the error probability is such that,

$$P_{error} = P(\varepsilon)$$

As a consequence of the Shannon theorem, when the outage event occurs, the system is in error almost surely, which means that the outage region is necessarily included in the error region, *i.e.*,

$$O \subseteq \varepsilon$$

For a given MIMO system operating at a rate R and under a power constraint P, the error probability is always lower-bounded by the outage probability as shown in Figure 2.3.2., *i.e.*,

$$\mathrm{P_{error}} \geq \mathrm{P_{out}}$$

The main objective is to design a coding scheme having an error probability that approaches the outage probability.

2.4. Design challenges without CSIT

In order to exploit fully the available diversity and the multiplexing gain of the MIMO channel, two approaches described in Paragraphs 2.4.1. and 2.4.2. have been considered in the literature. For these two approaches, the transmitted signal are coded across the time (during T time-slots) and the space (over the n_t antennas) using the so-called space-time coding. The transmitted codeword matrix $\mathbf{X} \in \mathbb{C}^{n_t \times T}$ is carved from a codebook denoted $\mathcal{X}_p$ having a size $|\mathcal{X}_p|$ and a rate R bits per channel use (bpcu) that is equal to

$$R = \frac{1}{T}\log_2|\mathcal{X}_p|.$$

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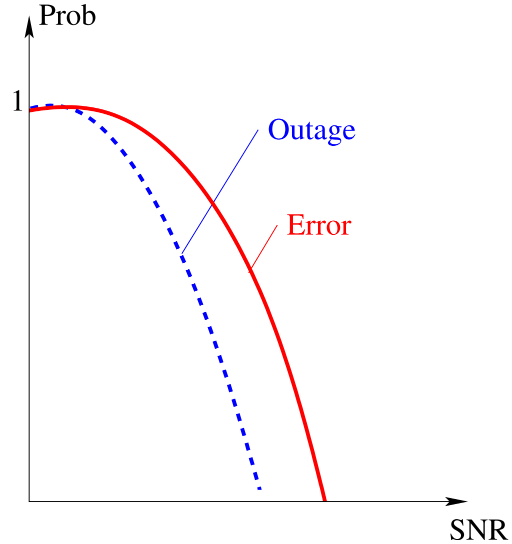

Figure 4: Relationship between outage and error event

2.4.1. Fixed rate code construction

The first approach proposed by Tarokh *et al.* in [2] considers the case of fixed data rate R that does not scale as SNR. This approach is more tailored to the Rayleigh fading distribution and consists in the minimization of the error probability over all the fading distribution to approach the outage probability behavior in the high SNR regime.

For a fixed rate R and using the eigenvalue distribution of the Wishart matrix $\mathbf{HH}^\dagger$, it is well known from that the outage probability scales in the high SNR regime as,

$$\mathrm{P}_{\mathrm{out}}(\mathrm{SNR}) \doteq \mathrm{SNR}^{-d}, \qquad \text{where } d = n_t \times n_r \tag{4}$$

The slope d of the outage probability $\mathrm{P}_{\mathrm{out}}(\mathrm{SNR})$ is called diversity gain. Notice that the number of independent paths for a $n_t \times n_r$ MIMO channel is also equal to $d = n_t n_r$. It can be then deduced that the maximal diversity gain of a MIMO system corresponds simply to the number of independent paths.

For a given channel realization, the error probability is upper-bounded using the union bound [2, 15] such that,

$$\mathrm{P}_{\mathrm{error}|\mathbf{H}} \leq \frac{1}{|\mathcal{X}_p|} \sum_{i,j:i\neq j} \mathrm{Prob}\{\mathbf{X}_i \to \mathbf{X}_j | \mathbf{H}\}$$

where $\mathrm{Prob}\{\mathbf{X}_i \to \mathbf{X}_j|\mathbf{H}\}$ denotes the pairwise error probability (PEP) between two codewords $\mathbf{X}_i$ and $\mathbf{X}_j$. The PEP refers to the probability that a certain true codeword $\mathbf{X}_i \in \mathcal{X}_p$ is mistaken with another codeword $\mathbf{X}_i$ assuming that these two codewords are the only codewords of the codebook. At high SNR, the most significant error event corresponds to the case of neighboring $\mathbf{X}_i$ and $\mathbf{X}_j$, i.e.,

$$\mathrm{P}_{\mathrm{error}|\mathbf{H}} \dot{\leq} \frac{1}{|\mathcal{X}_p|} \max_{i,j:i\neq j} \mathrm{Prob}\{\mathbf{X}_i \to \mathbf{X}_j | \mathbf{H}\}.$$

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Finally, the upper-bound on the error probability can be deduced by averaging over all the channel distribution. The goal of the space time code design is to minimize the worst case of error probability to approach the outage probability behavior in the high SNR regime.

2.4.2. Approximately universal code construction

While the first approach is more tailored to the Rayleigh fading distribution, Zheng and Tse proposed in [3] a powerful approach based on the high SNR characterization of the dual benefits in term of diversity and spatial multiplexing using the diversity multiplexing tradeoff (DMT) framework.

Definition 4. *Given a point-to-point MIMO system, the gains in terms of diversity gain d*

$$-d = \lim_{\mathrm{SNR}\to\infty} \frac{\log P_{out}(R,\mathrm{SNR})}{\log \mathrm{SNR}}$$

and spatial multiplexing gain r

$$r = \lim_{\mathrm{SNR}\to\infty} \frac{C(\mathrm{SNR})}{\log \mathrm{SNR}}$$

can be simultaneously obtained. But, there is a fundamental tradeoff $d_{\mathrm{out}}(r)$ between these two gains provided by any coding scheme.

Theorem 3 (Outage DMT of the MIMO channel)**.** *The DMT of $n_t \times n_r$ Rayleigh channel is a piecewise-linear function connecting the points $\big(r, d(r)\big)$ where $r = 0, \ldots, \min(n_t, n_r)$ and*

$$d(r) = (n_t - r)(n_r - r). \tag{5}$$

Definition 5. *A coding scheme $\mathcal{X}_p(\mathrm{SNR})$ with data rate R bits per channel use achieves a multiplexing gain r and diversity gain d if the data rate R is such that*

$$\lim_{\mathrm{SNR}\to\infty} \frac{R(\mathrm{SNR})}{\log \mathrm{SNR}} = r,$$

and the average error probability $P_e(\mathrm{SNR})$ with maximum likelihood-decoding is such that

$$-d = \lim_{\mathrm{SNR}\to\infty} \frac{\log P_e(\mathrm{SNR})}{\log \mathrm{SNR}}.$$

For a given multiplexing gain r, the largest diversity supported by any coding scheme is denoted by $d_{\mathcal{X}_p}(r)$.

For any coding scheme with rate scaling as $r \log \mathrm{SNR}$, the DMT of the code $d_{\mathcal{X}_p}(r)$ is upper bounded by $d_{\mathrm{out}}(r)$, i.e,

$$d_{\mathcal{X}_p}(r) \le d_{\mathrm{out}}(r). \tag{6}$$

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3. Code construction for flat fading channel

In this section, we consider the case of a flat fading MIMO channel where the wireless channel remains constant during all the duration of the transmission. The data bits streams are assumed to be transmitted over this wireless channel without using any outer codes.

3.1. Flat fading channel model

We consider first the case of flat fading channel model depicted in Figure 5 that is given by

$$\mathbf{Y} = \sqrt{\frac{\mathrm{SNR}}{n_t}}\,\mathbf{H}\mathbf{X} + \mathbf{Z} \tag{7}$$

where $\mathbf{X} \in \mathbb{C}^{n_t \times T}$ is a space time code drawn from code $\mathcal{X}_p$ of rate R per channel use, $\mathbf{Y} \in \mathbb{C}^{n_r \times T}$ is the received signal, $\mathbf{Z} \in \mathbb{C}^{n_r \times T} \sim \mathcal{CN}(0,1)$ is the additive noise and $\mathbf{H} \in \mathbb{C}^{n_r \times n_t}$ is the channel matrix with i.i.d complex Gaussian $\mathcal{CN}(0,1)$ entries. The scaling factor θ is chosen to ensure the power constraint,

$$\mathbb{E}\left[\|\mathbf{X}\|_{\mathrm{F}}^2\right] = T. \tag{8}$$

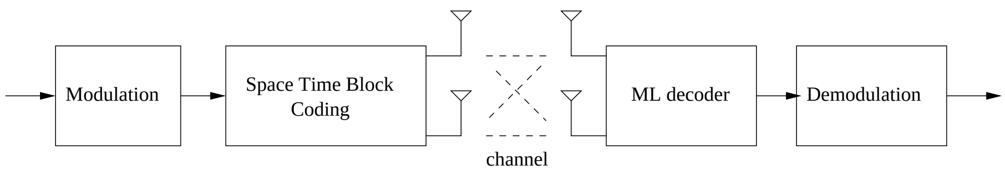

Figure 5: MIMO system

3.2. Optimal Maximum Likelihood (ML) MIMO decoder

For the SISO channel considered in Paragraph 2.1.1., the maximum likelihood decoder should find the QAM constellation that minimizes,

$$\hat{x} = \arg \min_{x \in \mathrm{QAM}} |y - hx|^2$$

The maximum likelihood decoder is also extended to the MIMO case. In this case, the receiver should find the matrix $\hat{\mathbf{X}}$ in the family of space time code that minimizes the following Frobenius norm,

$$\hat{\mathbf{X}} = \arg \min_{\mathbf{C} \in \mathcal{X}_p} \mathrm{Tr}[(\mathbf{Y} - \mathbf{HC})(\mathbf{Y} - \mathbf{HC})^\dagger]$$

Exhaustive research among all the possible space time codewords can be performed. Algorithm with less complexity such as sphere decoder [11] and Schnorr-Euchner [12] are used in practice.

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3.3. Space time code properties with fixed rate

In this section, we focus on space time codes design when the rate of the code is independent of SNR, *i.e.* $R(\mathrm{SNR}) = R$.

3.3.1. Error probability upper-bound

In this case, minimizing the average error probability over the distribution of the fading channel is studied. The average PEP for the $n_t \times n_r$ MIMO channel has been derived in [2].

Theorem 4 (PEP upper-bound)**.** *Assuming that a maximum likelihood decoder is used, the worst PEP is bounded by,*

$$\mathrm{PEP} \leq c\ \mathrm{SNR}^{-d}, \tag{9}$$

where d is the diversity given by,

$$d = n_r \,\mathrm{rank}\{\Delta\mathbf{X}\Delta\mathbf{X}^{\dagger}\},$$

and c is the coding gain and is equal to,

$$c = 4^{d} \left(\min_{\Delta\mathbf{X}\neq 0} \det\left\{\Delta\mathbf{X}\Delta\mathbf{X}^{\dagger}\right\}\right)^{-n_r}.$$

The upper-bound of $c\,\mathrm{SNR}^{-d}$ *is minimized if:*

(i)- The diversity order $d = n_r \,\mathrm{rank}(\boldsymbol{\Delta X}\boldsymbol{\Delta X}^{\dagger})$ *is maximized;*

(ii)- The coding gain $c = 4^{d}\left(\min_{\Delta\mathbf{X}\neq 0}\det\left\{\Delta\mathbf{X}\Delta\mathbf{X}^{\dagger}\right\}\right)^{-n_r}$ *is minimized.*

3.3.2. Code design criteria

As a consequence of the outage probability lower-bound, the maximal diversity order that can be achieved by a coding scheme is equal to $n_t \times n_r$. As shown in Theorem 4, the diversity order of the coding scheme when using an ML decoder is $n_r \times \mathrm{rank}(\boldsymbol{\Delta X}\boldsymbol{\Delta X}^{\dagger})$. This means that the diversity corresponding to the multiple receive antennas of n_r, known as receive diversity, can be achieved regardless the used coding scheme. However, in order to achieve full transmit diversity, the rank of $\Delta\mathbf{X}\Delta\mathbf{X}^{\dagger}$ should be equal to n_t.

The gain in diversity is illustrated in Figure 6 where it is shown that significant gains can be observed when increasing the slope of the error probability. Moreover, the minimization of the coding gain in Theorem 4 results on a left shift of the error curve in Figure 6.

An optimal linear space time code should satisfy thus the following design criteria:

- Full rate symbol = $\min(n_t, n_r)$
- Full diversity, *i.e.* $\mathrm{rank}\{\Delta\mathbf{X}\Delta\mathbf{X}^{\dagger}\} = n_t$.
- Minimized coding gain $c = 4^{d}\left(\min_{\Delta\mathbf{X}\neq 0}\det\left\{\Delta\mathbf{X}\Delta\mathbf{X}^{\dagger}\right\}\right)^{-n_r}$.

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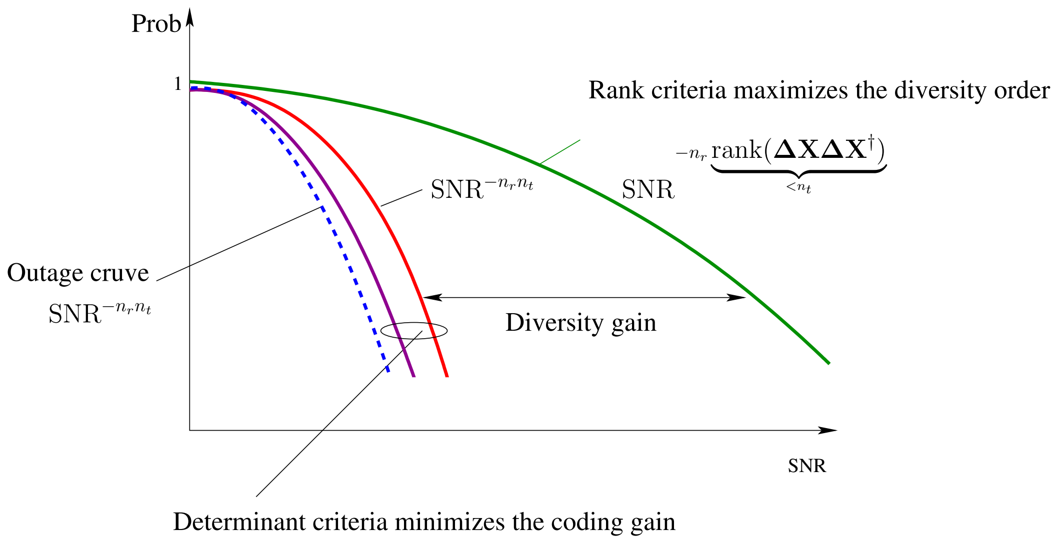

Figure 6: Rank and determinant criteria

3.3.3. Examples of space-time codes

To illustrate this, we consider a MIMO channel having $n_t = 2$ antennas at the transmitter side and $n_r = 2$ antennas at the receiver side. We assume that the channel is known only at the receiver side but not at the transmitter side. The binary information $b_1 b_2 \ldots$ are first modulated using a 2^m-QAM constellation $x_1 x_2 x_3 x_4 \ldots$. Then, the stream of data symbols is coded using a $n_t \times T$ space time coding before being transmitted on the MIMO channel as described in the channel model in (7).

The QAM symbols are carved from a normalized 2^m-QAM constellation that are scaled to match $\frac{1}{\sqrt{E_s}}\left[2\mathbb{Z}[i] + (1+i)\right]$ with

$$E_s = \frac{2(2^m - 1)}{3}.$$

It can be easily checked that the minimal distance between two neighboring symbols is $d_{\min} = \frac{2}{\sqrt{E_s}}$.

A. Spatial division multiplexing:

This method is also known as Vertical Bell Labs Space Time code (VBLAST) [13]. In this case, two different symbols are transmitted on the different antennas without any coding at each time slot. The corresponding codeword is therefore,

$$\mathbf{X} = \frac{1}{\sqrt{2}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Note that the normalization factor $\frac{1}{\sqrt{2}}$ is required in order to satisfy the power constraint in (8). For this code, two symbols are transmitted during one time slot and hence the symbol

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rate is 2 and is equal to the maximal multiplexing gain $\min(n_t, n_r) = 2$. The diversity of this code is,

$$d = n_r \, \text{rank}\{\Delta\mathbf{X}\Delta\mathbf{X}^\dagger\} \text{ with } \Delta\mathbf{X} \neq \mathbf{0}.$$

The difference codeword matrix is,

$$\Delta\mathbf{X} = \mathbf{X} - \mathbf{X}' = \frac{1}{\sqrt{2}} \begin{bmatrix} x_1 - x_1' \\ x_2 - x_2' \end{bmatrix}$$

is a rank one matrix. The diversity of this code is then only equal to 2. Finally, the minimal determinant of the SDM code is,

$$\min_{\Delta\mathbf{X}\neq\mathbf{0}} \det(\Delta\mathbf{X}\Delta\mathbf{X}^\dagger) = \min_{x_1 \neq x_1'} \frac{1}{2}|x_1 - x_1'|^2 = \frac{1}{2} d_{\min}^2 = \frac{2}{E_s}$$

and the PEP is upper-bounded by, The maximal error probability is therefore,

$$\boxed{\text{PEP} \leq 4E_s^2 \, \text{SNR}^{-2}.}$$

B. Repetition code :

For the repetition code described in [15], each data symbol is repeated during two time slots and the corresponding codeword is,

$$\mathbf{X} = \begin{bmatrix} x_i & 0 \\ 0 & x_i \end{bmatrix}.$$

Note that in this case, the power constraint in (8) is satisfied without need to multiply by a normalization factor as for the SDM case. It can be observed that for the repetition code only one symbol is transmitted during $T = 2$ time slots and then the symbol rate is $1/2$ which is less than the maximal multiplexing gain of 2. The diversity of this code can be computed from Theorem 4 as,

$$d = n_r \, \text{rank}\{\Delta\mathbf{X}\Delta\mathbf{X}^\dagger\} \text{ with } \Delta\mathbf{X} \neq \mathbf{0}.$$

The difference codeword matrix is,

$$\Delta\mathbf{X} = \mathbf{X} - \mathbf{X}' = \begin{bmatrix} x_1 - x_1' & 0 \\ 0 & x_1 - x_1' \end{bmatrix}.$$

and its determinant is

$$\det(\Delta\mathbf{X}\Delta\mathbf{X}^\dagger) = |x_1 - x_1'|^4 \neq 0 \text{ if } x_1 \neq x_1',$$

The non zero matrix is therefore full rank and the diversity that can be extracted is 4. The code extracts all the diversity of the 2×2 MIMO system but not all the degrees of freedom. The minimum of $\det(\Delta\mathbf{X}\Delta\mathbf{X}^\dagger)$ is,

$$\min_{\Delta\mathbf{X}\neq\mathbf{0}} \det(\Delta\mathbf{X}\Delta\mathbf{X}^\dagger) = \min_{x_1 \neq x_1'} |x_1 - x_1'|^4 = d_{\min}^4 = \frac{2^4}{E_s^2}.$$

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The error probability is then upper-bounded by,

$$\mathrm{PEP}_{\max} = c\,\mathrm{SNR}^{-4},$$

where

$$c = 4^{d}(\min_{\Delta\mathbf{X}\neq\mathbf{0}} \det(\Delta\mathbf{X}\Delta\mathbf{X}^{\dagger}))^{-n_r} = E_s^4.$$

The maximal error probability is therefore,

$$\boxed{\mathrm{PEP} \leq E_s^4\,\mathrm{SNR}^{-4}.}$$

C. Alamouti code

The Alamouti code structure in [14] is given by,

$$\mathbf{X} = \frac{1}{\sqrt{2}} \begin{bmatrix} x_1 & -x_2^* \\ x_2 & x_1^* \end{bmatrix}$$

Notice that the normalization factor of $\frac{1}{\sqrt{2}}$ is also required for the Alamouti case to satisfy the power constraint in (8). For the Alamouti code, two different symbols are transmitted during 2 time slots. The symbol rate of this code is then equal to 1 symbol per time slot which is not optimal in term of its multiplexing gain for the 2×2 MIMO case. However, for the 2×1 MISO configuration, this code is full rate. The diversity of this code can be computed by,

$$d = n_r\,\mathrm{rank}\{\Delta\mathbf{X}\Delta\mathbf{X}^{\dagger}\} \text{ with } \Delta\mathbf{X} \neq \mathbf{0}.$$

where

$$\Delta\mathbf{X} = \mathbf{X} - \mathbf{X}' = \frac{1}{\sqrt{2}} \begin{bmatrix} x_1 - x_1' & -(x_2 - x_2')^* \\ x_2 - x_2' & (x_1 - x_1')^* \end{bmatrix}.$$

The determinant of $\Delta\mathbf{X}\Delta\mathbf{X}^{\dagger}$ is,

$$\det(\Delta\mathbf{X}\Delta\mathbf{X}^{\dagger}) = \frac{1}{2}(|x_1 - x_1'|^2 + |x_2 - x_2'|^2)^2 \neq 0 \text{ if } (x_1, x_2) \neq (x_1', x_2')$$

This means that $\mathrm{rank}\{\Delta\mathbf{X}\Delta\mathbf{X}^{\dagger}\} = 2$ for $\Delta\mathbf{X} \neq \mathbf{0}$. The diversity achieved by this code is,

$$d = 2\times 2 = 4.$$

For 2×2 MIMO system, the Alamouti code extracts the diversity gain but not the full multiplexing gain. The minimum of $\det(\Delta\mathbf{X}\Delta\mathbf{X}^{\dagger})$ is,

$$\min_{\Delta\mathbf{X}\neq\mathbf{0}} \det(\Delta\mathbf{X}\Delta\mathbf{X}^{\dagger}) = \frac{1}{2} \min_{x_1\neq x_1', x_2\neq x_2'} (|x_1 - x_1'|^2 + |x_2 - x_2'|^2)^2 = \frac{1}{2} d_{\min}^4 = \frac{2^3}{E_s^2}.$$

The PEP is then upper-bounded by,

$$\mathrm{PEP} \leq c\,\mathrm{SNR}^{-4},$$

where

$$c = 4^{d}(\min_{\Delta\mathbf{X}\neq\mathbf{0}} \det(\Delta\mathbf{X}\Delta\mathbf{X}^{\dagger}))^{-n_r} = 4E_s^4.$$

The maximal error probability is therefore,

$$\boxed{\mathrm{PEP} \leq 4E_s^4\,\mathrm{SNR}^{-4}.}$$

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D. Golden code

This code was introduced in [5]. The codeword matrix of the Golden code $\mathbf{X} = \frac{1}{\sqrt{2}} \times \boldsymbol{C}$ where,

$$\boldsymbol{C} = \frac{1}{\sqrt{5}} \begin{bmatrix} \alpha(x_1 + \theta x_2) & \bar{\alpha}(x_3 + \bar{\theta} x_4) \\ i\alpha(x_3 + \theta x_4) & \bar{\alpha}(x_1 + \bar{\theta} x_2) \end{bmatrix},$$

and $\theta = \frac{1+\sqrt{5}}{2}$, $\bar{\theta} = \frac{1-\sqrt{5}}{2}$, $\alpha = 1 + i - i\theta$ and $\bar{\alpha} = 1 + i - i\bar{\theta}$.
We can verify that $\theta\bar{\theta} = -1$ and $\alpha\bar{\alpha} = 2 + i$. This means that using the Golden code the combined symbols sent at each antenna has energy equal to 1 and therefore a normalization factor of $\frac{1}{\sqrt{2}}$ is required to satisfy (8). Using the Golden code structure, it can be seen that four symbols are transmitted during $T = 2$ time slots. The symbol rate is therefore equal to 2. The diversity of this code can be computed by,

$$d = n_r \, \mathrm{rank}\{\Delta\mathbf{X}\Delta\mathbf{X}^\dagger\} \text{ with } \Delta\mathbf{X} \neq \mathbf{0}.$$

The difference codeword matrix is given by,

$$\Delta\boldsymbol{C} = \boldsymbol{C} - \boldsymbol{C}' = \frac{1}{\sqrt{5}} \begin{bmatrix} \alpha((x_1 - x_1') + \theta(x_2 - x_2')) & \bar{\alpha}((x_3 - x_3') + \bar{\theta}(x_4 - x_4')) \\ i\alpha((x_3 - x_3') + \theta(x_4 - x_4')) & \bar{\alpha}((x_1 - x_1') + \bar{\theta}(x_2 - x_2')) \end{bmatrix}.$$

The minimal determinant of this matrix is $\min_{\Delta\boldsymbol{C}\neq 0} \det(\Delta\boldsymbol{C}) = \frac{2+1i}{(\sqrt{5})^2}$ and

$$\min_{\Delta\boldsymbol{C}\neq 0} \det(\Delta\boldsymbol{C}\Delta\boldsymbol{C}^\dagger) = \frac{|2+1i|^2}{25} = \frac{1}{5}.$$

The matrix is therefore full rank and the maximal diversity is 4. The Golden code extracts the full diversity of the 2×2 MIMO system and the full multiplexing gain. The error probability is upper-bounded by,

$$\mathrm{PEP} \leq c\,\mathrm{SNR}^{-4},$$

where

$$c = 4^d \left(\min_{\Delta\mathbf{X}\neq\mathbf{0}} \det(\Delta\mathbf{X}\Delta\mathbf{X}^\dagger) \right)^{-n_r} = 100\, E_s^4.$$

The maximal error probability is,

$$\boxed{\mathrm{PEP} \leq 100\, E_s^4\, \mathrm{SNR}^{-4}.}$$

D. Codes comparison:

In order to have a fair comparison, the four above schemes should be compared using the same bit rate, for example 4 bits per time slot. As we can see from the above discussion, these codes have different symbol rate. In order to achieve the same bit rate per channel use, different 2^m-QAM constellation should be used to map the symbols of the different space time coding schemes. The VBLAST code and the Golden code have a symbol rate of 2. The bit rate when symbols are mapped using 2^m-QAM constellation is $2m$ and is equal to 4 if $m = 2$. Symbols of these two schemes should be then mapped using QPSK constellation.

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The repetition code has a symbol rate of $1/2$. The bit rate of this code when using 2^m-QAM constellation is $1/2m$ and is equal to 4 if $m = 8$. This corresponds then to a 256QAM constellation. The Alamouti code has a symbol rate of 1. Using a 2^mQAM constellation, the bit rate of m is equal to 4 if $m = 4$ which corresponds to a 16QAM constellation.

Table 1 summarizes the PEP upper-bound of the schemes with their corresponding constellation. As it can be seen from Table 1, the upper-bound on the pairwise error probability expression depends on the choice of the constellation. constellation.

Space Time Code	**Constellation**	**PEP upper-bound**
VBLAST	QPSK	16 SNR^{-2}
Repetition code	256QAM	170^4 SNR^{-4}
Alamouti code	16QAM	4 10^4 SNR^{-4}
Golden code	QPSK	400 SNR^{-4}

Table 1. Upper-bound on PEP for a bit rate of 2 bpcu.

The error performances of these space time code are illustrated in Figure 7. At very low

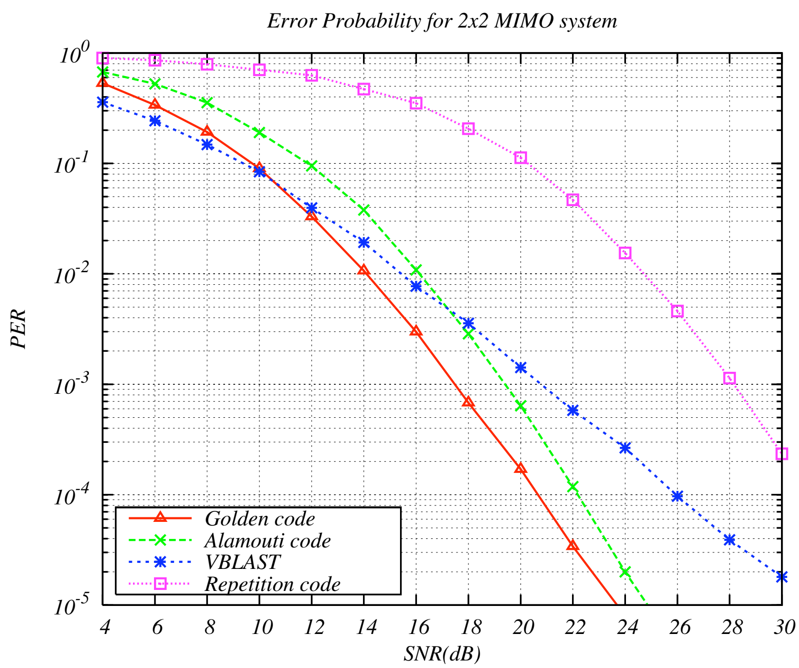

Figure 7: Comparison between error probability for the 2×2 MIMO scheme with a bit rate of 2 bpcu.

SNR, the SDM has better coding gain then both Alamouti schemes and the Golden code (Table 1) and has a better PER than the Golden code and the Alamouti code. However, for the high SNR regime, the diversity gain dominates the coding gain. As the Alamouti scheme is not a full rate scheme, using a 16QAM constellation instead of QPSK constellation in order to achieve the same spectral efficiency induces a loss in term of coding gain. The

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gain of the Golden code in diversity and in coding gain can be also observed in Figure 7. The repetition code has a full diversity of 4. However its coding gain in Table 1 is very high as a 256QAM constellation is used to compensate the loss in the rate symbol. This high coding gain dominates the error probability in the low SNR regime and the repetition code performs badly compared to the SDM and the other codes (Figure 7). In the high SNR regime, the gain diversity of the code becomes more dominant and the slope of the error probability becomes higher and compensate the loss in SNR compared to the SDM case.

3.4. Approximately universal code over a flat fading channel

The approximately universal code design provides a structured code design criterion that achieve the DMT [6]. This design criteria is derived from the performance of the code over the worst-channel case that is not in outage. Universal codes achieve reliable communication over MIMO channel realization that are not in outage. In the following, we review from [5, 6] the non-vanishing design criteria and the DMT-achieving construction of the perfect space time code in [4].

3.4.1. Sufficient condition for DMT achievability

Theorem 5. *A coding scheme $\mathcal{X}_p(\mathrm{SNR})$ is approximately universal over the MIMO channel if and only if, for every pair of distinct codewords*

$$\mu_1^2\mu_2^2\ldots\mu_n^2 \geq \frac{c}{2^{R(\mathrm{SNR})+o(\log \mathrm{SNR})}}, \quad c>0 \tag{10}$$

where $n=\mathrm{rank}\{\mathbf{HH}^\dagger\}=\min(n_t,n_r)$ and $\mu_1^2\leq\ldots\leq\mu_n^2$ are the m eigen-values of the codewords difference matrix $\Delta\mathbf{X}\Delta\mathbf{X}^\dagger$.

3.4.2. Non Vanishing Determinant (NVD) code definition

The non-vanishing determinant (NVD) criteria is a particular form of the approximately universal condition defined in Theorem 5. Before defining the NVD code structure in [4], we first define the normalized space time code $\mathbf{X}$ by,

$$\mathbf{X}=\theta\bar{\mathbf{X}}, \tag{11}$$

where $\bar{\mathbf{X}}\in\bar{\mathcal{X}}_p(\mathrm{SNR})$ refers to the normalized space time code and θ is the scaling factor that ensures the power constraint in (8).

Definition 6 (NVD codes)**.** *A coding scheme $\bar{\mathcal{X}}_p(\mathrm{SNR})$ is called a rate-n NVD code if $\mathcal{X}_p(\mathrm{SNR})$ satisfies the following properties*

- *Each entry $\bar{x}_{i,j}$ of the $\mathcal{X}_p(\mathrm{SNR})$ is a linear combination of symbols from $\mathcal{A}(\mathrm{SNR})$, where $\mathcal{A}(\mathrm{SNR})$ is a universal code over the scalar channel with data rate $R_{\mathcal{A}}(\mathrm{SNR})$ bits PCU. The quadrature amplitude modulation (QAM) constellation such as QPSK, 16QAM, 64QAM,... or HEX constellation are usually used as scalar universal codes.*

- *The average number of symbols transmitted by $\mathcal{X}_p(\mathrm{SNR})$ is equal to n symbols per channel use.*

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- *The following NVD property is satisfied*

$$\det\left(\Delta\bar{\mathbf{X}}\Delta\bar{\mathbf{X}}^{\dagger}\right) \dot{\geq} \text{SNR}^{0} \tag{12}$$

3.4.3. Scaling factor θ for NVD codes

For a non-vanishing determinant code, the average number of symbols transmitted by $\mathcal{X}_p(\text{SNR})$ is $n = \min(n_t, n_r)$ symbols per channel use and can be defined as

$$n = \frac{1}{T}\log_{|\mathcal{A}|}|\mathcal{X}_p|, \tag{13}$$

where $|\mathcal{X}_p|$ denotes the total number of possible codewords in $\mathcal{X}_p$ and $|\mathcal{A}|$ denotes the total number of constellation symbols in $\mathcal{A}$. Equivalently,

$$|\mathcal{X}_p| = |\mathcal{A}|^{nT}. \tag{14}$$

Let $\mathcal{A}$ be the 2^mQAM constellation[1] with size $|\mathcal{A}| = 2^m$, such that

$$\mathcal{A} = \left\{a + ib, \quad |a|, |b| \leq m-1 \quad a, b \text{ are odd} \right\}.$$

The rate of the space time code $R = r\log\text{SNR}$ can be related to $|\mathcal{X}_p|$ by $R = \frac{1}{T}\log_2|\mathcal{X}_p| = r\log\text{SNR}$. or equivalently

$$|\mathcal{X}_p| = \text{SNR}^{Tr}. \tag{15}$$

By combining (14) and (15), it follows that,

$$|\mathcal{A}| = \text{SNR}^{\frac{r}{n}}. \tag{16}$$

As each entry $\bar{x}_{i,j}$ is a linear combination of symbols s_l carved from a 2^m-QAM constellation, i.e

$$\bar{x}_{i,j} = \sum_{l=1}^{n} a_l s_l, \quad a_l \in \mathbb{C} \text{ and } \|\mathbf{a}\|^2 = \|[a_1 \ldots a_n]\|^2 = 1$$

then it can be easily checked that ,

$$\mathbb{E}[|\bar{x}_{i,j}|^2] = \|a\|^2\,\mathbb{E}[|s|^2] = E_s = \frac{2(|\mathcal{A}|-1)}{3} \doteq \text{SNR}^{\frac{r}{n}}.$$

Using the normalization constraint in (8), it follows that

$$\theta^2 \doteq \text{SNR}^{-\frac{r}{n}}. \tag{17}$$

3.4.4. DMT of the code

Lemma 1. *NVD codes achieve the DMT for $n_t \times n_r$ MIMO configuration when $n_t \leq n_r$, and for full rate codes ($n = n_t$).*

Proof. The determinant of the non normalized space time code is such that

$$\det\left(\Delta\mathbf{X}\Delta\mathbf{X}^{\dagger}\right) = \theta^{2n_t}\det\left(\Delta\bar{\mathbf{X}}\Delta\bar{\mathbf{X}}^{\dagger}\right).$$

By replacing the scaling factor θ^2 by its value in (17) with $n = n_t$, the NVD property is then satisfied *i.e.*,

$$\det\left(\Delta\mathbf{X}\Delta\mathbf{X}^{\dagger}\right) \dot{\geq} \text{SNR}^{-r}\,\text{SNR}^{0} \doteq \frac{c}{2^{R(\text{SNR})+o(\text{SNR})}}.$$

□

[1]The same value of scaling factor is valid using HEX constellation.

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3.4.5. NVD code construction: Perfect space time codes

Perfect space time codes are full rate codes ($n = n_t$) constructed from cyclic division algebras (CDA) defined as following. Let $\mathbb{L} = \mathbb{Q}(i,\theta)$ be a cyclic extension of degree n_t on the base field $\mathbb{Q}(i)$. The generator of Galois group $\mathrm{Gal}(\mathbb{L}/\mathbb{Q}(i))$ is denoted by σ, and assume that $\mathrm{Gal}(\mathbb{L}/\mathbb{Q}(i)) = \{\sigma^0,\ldots,\sigma^{n_t-1}\}$. Let $\gamma \in \mathbb{Q}(i)$ be such that $\gamma,\gamma^2,\ldots,\gamma^{n_t-1}$ are non-norm elements in $\mathbb{L}$. The CDA of degree n_t is given by

$$\mathcal{C} = \left(\mathbb{L}/\mathbb{Q}(i),\sigma,\gamma\right).$$

Each element $\bar{\mathbf{X}}$ of $\mathcal{C}$ is given by,

$$\bar{\mathbf{X}} = \begin{pmatrix} x_1 & x_2 & \ldots & x_{n_t} \\ \gamma\sigma(x_{n_t}) & \sigma(x_1) & \ldots & \sigma(x_{n_t-1}) \\ \vdots & & & \vdots \\ \gamma\sigma^{n_t-1}(x_2) & \gamma\sigma^{n_t-1}(x_3) & \ldots & \sigma^{n_t-1}(x_1) \end{pmatrix} \tag{18}$$

where $x_i \in \mathbb{I} \subset \mathcal{O}_{\mathbb{L}}$ is a linear combination of symbols carved from a QAM or Hex constellation, $\mathcal{O}_{\mathbb{L}}$ being the ring of the integers, and $\mathbb{I}$ is an properly chosen ideal that preserves the constellation shaping. As perfect space time codes are linear codes constructed from a CDA, then

$$\min_{\Delta\bar{\mathbf{X}}\neq 0} \det\{\Delta\bar{\mathbf{X}}\Delta\bar{\mathbf{X}}^\dagger\} \geq \delta,$$

where δ is the inverse of the discriminant of $\mathbb{Q}(\theta)$ (refer to [4] for more details), and is independent of the constellation size. The NVD property in (12) can be easily verified, such that the determinant of the non-normalized space time code is such that,

$$\det\left(\Delta\mathbf{X}\Delta\mathbf{X}^\dagger\right) \geq \frac{\delta}{2^{R(\mathrm{SNR})+o(\mathrm{SNR})}}.$$

4. Practical insights on the use of MIMO diversity techniques

Unlike the simplified flat fading channel model considered in previous sections, the industrial transmission schemes are based on more realistic assumptions that include the use of the bit interleaved coded modulation scheme and a multi-tap channel. This section will review the result in [10] where the impact of concatenating multidimensional coding scheme with outer codes is studied.

4.1. BICM system model

The block diagram of the considered system based on the IEEE 802.11n transmission scheme is depicted in Figure 8. During the transmission, the binary information elements $\underline{b}$ are first encoded by a binary code of rate R_c *e.g.* a convolutional code $\mathcal{C}$ with free distance d_{free} and then interleaved by a bit interleaver π. The coded and interleaved sequence $\underline{c}$ is fed into the 2^m-QAM gray mapper and is mapped into a sequence of symbols $\underline{x}$. The resulting symbols are coded by a space time block codeword (STBC) $\mathcal{X}_p$ that associates to each $1 \times sr$ symbol vector a $n_t \times s$ matrix where s is the coding duration of the STBC known also as

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STBC spreading factor and r is the number of symbols transmitted during one time slot *i.e.*, the rate of the STBC[2]. The coded space-time codes are finally transmitted on a multiple antenna channel $\mathbf{H}$ with n_t transmit antennas and n_r receive antennas.

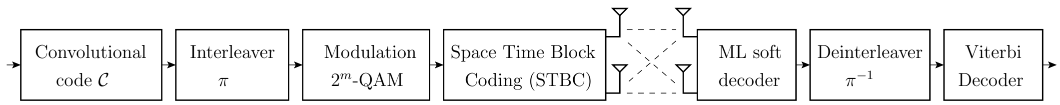

Figure 8: BICM MIMO system

Channel model

The two cases where the channel is either flat fading or selective in frequency are addressed. The flat fading channel models the case when the channel remains constant during all the duration of coding. For the flat fading channels, N space time block codewords are transmitted. Each STBC has a coding duration equal to s time slots. The second considered model is the frequency selective fading which models the case when the bandwidth of the transmitted signal exceeds the coherence bandwidth of the channel. For a frequency selective channel with a delay profile of L taps, a MIMO-OFDM system is considered. The channel is therefore decomposed into N parallel channels, $\mathbf{H}(f_k) = \mathbf{H}(e^{j2\pi f_k})$ with $k = 0 \ldots N-1$ that are statistically correlated where f_k is the channel frequency subcarrier. We assume that the channel is flat over each subcarrier f_k and that each space-time codeword is transmitted over one subcarrier f_k where $k = 0 \ldots N-1$. Finally, to simplify the notation for both cases of flat fading and frequency selective channel, we refer by k the indexing of the space time codewords that will be used also to index the channel realization over the k-th STBC, *i.e.*, $\mathbf{H}(k) = \mathbf{H}(sk + j)$ with $j = 0 \ldots s-1$.

Bit interleaver

Before detailing the interleaver structure, we recall that each block STBC is associated to a vector of $1 \times sr$ symbol vector or equivalently to a bit vector of $1 \times msr$.
The bit interleaver can be modeled as π: $k^{'} \to (k,i)$ where $k^{'}$ denotes the original ordering of the coded bits $c_{k'}$, k denotes the index of the STBC matrix and i indicates the position of the bits $c_{k'}$ in the corresponding $1 \times msr$ bit vector associated to the k-th STBC.

At the receiver, the received coded space time codeword is given by

$$\mathbf{Y}(k)^{[n_r \times s]} = \mathbf{H}(k)^{[n_r \times n_t]}\mathbf{C}(k)^{[n_t \times s]} + \mathbf{Z}(k)^{[n_r \times s]}, \tag{19}$$

where $\mathbf{Z}(k) \sim \mathcal{CN}(0, N_0 \mathbf{I}_{n_r})$ is the complex additive Gaussian noise.

ML soft decoder

The maximum likelihood (ML) soft decoder generates for each coded bit $c_{k,i}$ two metrics: $\lambda^i_{c_k=0}$ and $\lambda^i_{c_k=1}$. Theses metrics correspond to the log-MAP (Maximum A-Posteriori) com-

[2]We recall from [3] that $r \leq \min(n_t, n_r)$

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puted over one codeword (refer to [7] for more details on λ-metrics), and are given by:

$$\begin{aligned}\lambda^i(c_k) &= \log \sum_{\mathbf{C}\in\mathcal{X}^i_{c_k}} p(\mathbf{Y}(k)\big|\mathbf{H}(k),\mathbf{C}),\\ &= \log \sum_{\mathbf{C}\in\mathcal{X}^i_{c_k}} \exp^{-\|\mathbf{Y}(k)-\mathbf{H}(k)\mathbf{C}(k)\|^2_{\mathrm{F}}}.\end{aligned}$$

These metrics can be well approximated by

$$\lambda^i(c_k) \approx \min_{\mathbf{C}\in\mathcal{X}^i_{c_k}} \|\mathbf{Y}(k)-\mathbf{H}(k)\mathbf{C}\|^2_{\mathrm{F}}, \tag{20}$$

where $\mathcal{X}^i_b$ denotes the constellation subset $\mathcal{X}^i_b = \{\mathbf{C}\in\mathcal{X}_p : l^i(\mathbf{C}) = b\}$ and $l^i(\mathbf{C})$ is the i^{th} bit in the corresponding $1\times msr$ bit vector associated to $\mathbf{C}$. The problem in (20) is a variant of the well known closest problem and can be solved using low complexity algorithms such as accelerated decoder in [16]. Finally, λ metrics associated to the interleaved bits are de-interleaved and are used by the Viterbi decoder to decode the information bits by finding the shortest path in the trellis according to,

$$\underline{\hat{c}} = \arg\min_{c} \sum_{k'} \lambda(c^i_k). \tag{21}$$

4.2. MIMO gains for flat fading MIMO BICM channels

The first practical scenario is the case of MIMO-BICM system and a transmission over a flat fading channel. For this scenario, an upper-bound on the pairwise error probability was derived in [10] for both cases when MIMO symbols are coded with a perfect space time code and the case when there is no space time coding for the MIMO symbols *i.e.* the case of spatial division scheme (SDM).

In the first case when perfect space time codes are used, the PEP is upper-bounded at high SNR by,

$$\mathrm{PEP} \le \left(d_{\mathrm{free}}\delta\right)^{-n_r}\left(E_s\right)^{n_t n_r}\mathrm{SNR}^{-n_t n_r}. \tag{22}$$

On the other hand, when no space time code is used (SDM scheme), the PEP can be upper-bounded by

$$\mathrm{PEP} \le \left(d_{\mathrm{free}}\right)^{-n_r}\left(E_s\right)^{n_r}\mathrm{SNR}^{-n_r}. \tag{23}$$

From these PEP upper-bound, it can be easily seen that the perfect codes extract the full MIMO diversity $n_t n_r$. However, without MIMO coding (the spatial division schemes), the diversity is only n_r. To illustrate this, we consider the case of 2×2 MIMO system with QPSK constellation. The values of the upper-bounds in (22) and (23) can be written in the form $\gamma_{\mathrm{as}}\,\mathrm{SNR}^{-d}$ and are summarized in Table 3 for different values of free distance.

As it can be seen from Table 3, the asymptotical upper-bound gain γ_{as} of the SDM is lower than the one of the GC. However in the high SNR regime, this asymptotical upper-bound gain will not be significant and the diversity gain of the GC becomes more dominant. Two main remarks can be deduced: the first one is that at the high SNR the GC with a weak convolutional code gives better performance than the SDM with a strong outer code. The

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Coding scheme	PEP upper-bound
GC without CC	$400\,\text{SNR}^{-4}$
SDM without CC	$4\,\text{SNR}^{-2}$
GC with $d_{\text{free}} = 5$	$16\,\text{SNR}^{-4}$
SDM with $d_{\text{free}} = 5$	$0.16\,\text{SNR}^{-2}$
GC with $d_{\text{free}} = 10$	$4\,\text{SNR}^{-4}$
SDM with $d_{\text{free}} = 10$	$0.04\,\text{SNR}^{-2}$

Table 2. Theoretical upper-bound for a 2×2 flat fading MIMO-BICM with QPSK symbols

second remark is that the gain of the GC vs the SDM schemes can be observed at a moderate PEP range when no convolutional code is used. This will not be the case in a MIMO-BICM system and especially for higher values of d_{free}. Reaching very low target PER will be very hard to simulate in a real simulation context where reasonable target error rate are often addressed.

4.3. MIMO gains for frequency selective MIMO BICM channels

The second scenario studied in [10] is the case of BICM-MIMO system with frequency selective fading channel. When MIMO symbols over each subcarrier are coded using perfect space time codes, the PEP is upper-bounded at high SNR by,

$$\text{PEP} \leq \mathcal{G}\,\text{SNR}^{-n_r n_t \min(L,D)}, \tag{24}$$

where $\mathcal{G} = \left(\alpha\,\delta\right)^{-n_r d_{\text{free}}} (E_s)^{n_t n_r \min(L,D)}$ and $D \leq d_{\text{free}}$ denotes the number of different subcarriers on which erroneous bits are received. The interleaver design maximizes the parameter D. The parameter α is a constant that depends on the covariance matrix.

For the case, when no space time code is used, the PEP is bounded such that,

$$\text{P}(\underline{c} \to \underline{\hat{c}}) \leq \left(\alpha\right)^{-n_r d_{\text{free}}} (E_s)^{n_r \min(L,D)}\,\text{SNR}^{-n_r \min(L,D)}, \tag{25}$$

where α depends on the covariance matrix.

The numerical value of the upper-bounds in (24) and (25) are illustrated in Table 3 for the case of 2×2 MIMO with QPSK symbols over a frequency fading channel with $L = 6$ where the values of $D_1 \leq 5$ and $D_2 \leq 10$ depend on the used interleaver. The same remarks

Coding scheme	PEP upper-bound
GC with $d_{\text{free}} = 5$	$(5\alpha^{-1})^{10}\,(\text{SNR}/2)^{-4\min(6,D_1)}$
SDM with $d_{\text{free}} = 5$	$\alpha^{-10}\,(\text{SNR}/2)^{-2\min(6,D_1)}$
GC with $d_{\text{free}} = 10$	$(5\alpha^{-1})^{20}\,(\text{SNR}/2)^{-4\min(6,D_2)}$
SDM with $d_{\text{free}} = 10$	$\alpha^{-20}\,(\text{SNR}/2)^{-2\min(6,D_2)}$

Table 3. Theoretical upper-bound for a 2×2 MIMO with QPSK symbols over a frequency fading channel with $L = 4$.

as for the flat fading channel concerning the difficulty of observing the additional diversity

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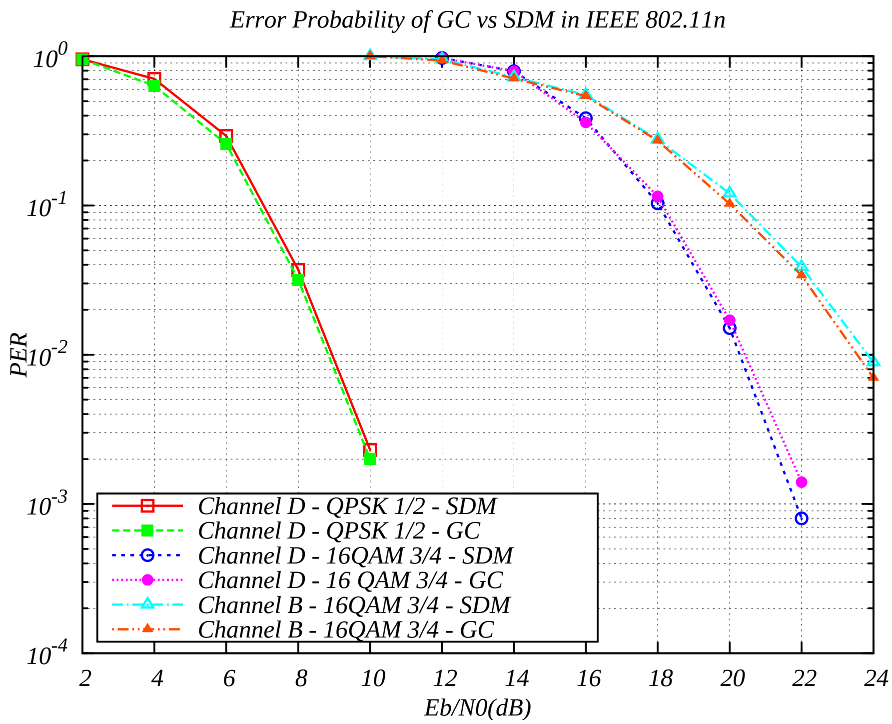

Figure 9: Golden Code vs SDM in IEEE 802.11n context

gain of the GC can be observed also here. Moreover, this will be become more difficult to observe than the flat fading channel as in this case both schemes gain in diversity.

The performance of the Golden code versus SDM has been evaluated in the IEEE 802.11n context in terms of packet error rate (PER) versus SNR, for a packet length of 1000-bits. In the following, SNR gain will be related to a PER of 10^{-2}. The packet error rates in Figure 9 are evaluated over channel D using QPSK and 16QAM constellation. The channel D is characterized by a $50ns$ rms delay spread and 18 taps, and then by significant frequency diversity. In the IEEE 802.11n context, the convolutional code $[133\ 171]$ with a coding rate of $R_c = 1/2$ is used with $d_{\text{free}} = 10$. No additional gain is observed at a PER $= 10^{-2}$. The channel B in the IEEE 802.11n standard can be assimilated to a flat fading channel, for which the additional gain using a convolutional code with high free distance ($d_{\text{free}} = 10$) cannot be observed at reasonable PER.

4.4. Practical limits of space time codes use in a standard context

Recent standards that use MIMO system such that IEEE 802.11n and IEEE 802.16e aim to increase the throughput and the reliability of the system. However, increasing the reliability comes often at the expense of increased complexity at both transmitter and receiver side. Scarifying the complexity order can be done if promising gains at reasonable PER range are observed. Although theoretically, using the upper-bound on the PEP at high SNR, one expects that the error probability of these codes should be shifted more to the left compared with the SDM case, practical assumptions are more realistic, and address generally a

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moderate SNR regime and moderate range of PER.

For the flat fading case, when no outer code is used, the huge gain observed by the Golden code over all other known code make it promising to be used in such systems. However, when a complete chain as BICM-MIMO-OFDM system is considered, the situation become considerably different. As we show in a BICM-MIMO-OFDM system, the diversity of BICM-OFDM system can be extracted when no space time code is used. Additional diversity can be provided by using perfect space time codes over each subcarriers. This additional diversity comes at the expense of an increased lattice decoder, *i.e.* instead of using a $2n_t \times 2n_t$ ML soft decoder a $2n_t^2 \times 2n_t^2$ is required. Moreover, the impact of this additional diversity cannot be unfortunately observed at moderate range of PER.

5. Concluding remarks

In this chapter, we reviewed the principle basis on the conception of multiple input multiple output systems. The knowledge of the channel knowledge at the transmitter side is crucial for this design. The case of a flat fading MIMO channel when the channel remains constant during all the duration of the transmission was firstly addressed. For the full CSIT case, we showed how the optimal power allocation over the eigen-channel modes optimizes the capacity of the channel. When no CSIT is available, we presented the diversity techniques used to benefit from the MIMO gains in term of diversity and multiplexing gain. The conception of these diversity techniques relies on two approaches. The first approach is based on the minimization of the pairwise error probability upper bound and yields to the well known space time code design criteria by Tarokh *et al.* (the rank and the determinant criteria) when the fading paths are assumed to be Rayleigh distributed. The second approach relies on the characterization of the diversity multiplexing tradeoff and is independent of the fading distribution. The construction of the perfect space time code family from the cycle division algebra that fully achieves these two gains is then detailed and the gain of the 2×2 MIMO Golden code are illustrated. Finally, we reviewed from [10] the impact of the concatenation of these multi-dimensional space time codes with powerful outer codes. The upper-bound on the PEP derived in [10] when using perfect space time codes is much lower than the one with space time codes. This should be traduced normally by a shift to the left of the packet error rate probability in the log-log scale. For the 2×2 MIMO channel, we show that these gains cannot be observed in the IEEE 802.11n context at a reasonable PER or SNR.

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