Extended Golden Light Code for FSO-MIMO Communications with Time Diversity

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***Abstract*—In this paper, we consider a $2 \times n_r$ Multiple Input Multiple Output - Free Space Optical (FSO-MIMO) channel with a time diversity order of two. We propose an Extended Golden Light Code (E-GLC) construction that encodes data over two independent optical channel realizations. The main objective of this code construction is to jointly extract the spatial and the temporal diversity while keeping a full rate of two symbols per channel use. The proposed coding scheme is compliant with optical Intensity Modulation and Direct Detection (IM/DD) constraints that limit the laser outputs to positive magnitude values with peak and average constraints. The error rate performance evaluations show that the E-GLC outperforms other known replication-based schemes.**

***Index Terms*—Optical Wireless Communication (OWC), Intensity Modulation (IM), MIMO system.**

I. INTRODUCTION AND MOTIVATIONS

OPTICAL carriers are considered as promising alternatives to support high data rate, large bandwidth within an unlicensed spectrum and high transmission security. In a Free Space Optical (FSO) system using Intensity Modulation and Direct Detection (IM/DD), the electrical signal is converted to an optical intensity that is projected via a laser beam to the optical receiver along the free space. This optical signal is detected at the receiver and converted back to an electrical signal. The main drawbacks of the FSO technology are the weather-dependent atmospheric turbulences and the pointing misalignment losses. These attenuations induce random fluctuations and degradation of the optical intensity at the receiver side and penalize the error rate performances. To mitigate these random fluctuations, referred in the following by fading, diversity transmission schemes are used [1, 2].

Deploying multiple lasers beams at the transmitter side and multiple apertures at the receiver side, also known as FSO-MIMO system, reduces significantly the error performance by exploiting the spatial diversity. To ensure uncorrelated fading, the spacing between lasers and receiver apertures should be higher than the spatial fading coherence length. In this paper, we limit the number of lasers to two and the receiver apertures to at least two. For this MIMO configuration, different space time coding schemes were proposed in literature to extract the spatial diversity order using full rate code as in [3, 4] or codes with symbol rate of one as the Repetition Code (RC) [5]–[8] or the Orthogonal Space Time Block Code (OSTBC) Alamouti Code [5].

The atmospheric turbulence in FSO channels is generally considered as a slow-time varying process [10]. The time diversity can be exploited considering burst-mode data transmission with frame duration at least equal to the twice duration of the coherence time [11]–[15]. To limit the delay-latency, we assume here a temporal diversity order of two where two independent channels realizations are observed within the frame transmission duration. To explore the time-diversity, error control coding schemes were proposed in [11] and interleaving is used to enhance the channel coding performances. Other solutions consist to re-transmit delayed signal version as in [12]–[14].

Increasing the spatial or/and the temporal diversity order comes at the expense of an increased complexity at the transmitter and receiver side and a high delay latency. This is due to the high optical spatial coherence length and coherence time. Exploiting jointly the spatial and the temporal diversity keeps a good tradeoff between latency, complexity and fading mitigation. Most of the literature proposed coding scheme are based on the replication of the spatial diversity scheme on the independent blocks as in [15]. Although these replication-based schemes extract the temporal and the spatial diversity, they induce a significant loss in the symbol rate.

Contributions: In this paper, we consider a $2 \times n_r$ MIMO configuration (with $n_r \geq 2$) with two block fading channels. Mono-chromatic lasers with multi-level Pulse Amplitude Modulation (PAM) are used to encode data bits as in [16]. The intensity at the lasers are constrained to be positive with peak and average constraints. Based on the Non Vanishing Determinant (NVD) criterion that guarantees that the full diversity order is achieved [22]–[24], we propose a full rate code construction with components in the real space $\mathbb{R}$. A bipolar to unipolar conversion is next performed to meet the positive, average and peak constraints. We show that, the E-GLC has an equivalent structure as the Golden Light Code [3, 4], when the symbols-space $\mathbb{Z}$ of this latter is extended to a two-dimensional space.

Outline: The rest of the paper is organized as follows. We start first by defining in Section II optical MIMO system characteristics, the spatial and temporal variations and the statistical distribution of the FSO channel. Considering the $2 \times n_r$ MIMO configuration and a time-diversity order of 2, we propose in Section III an Extended-Golden Light Code construction that jointly extracts the spatial and temporal diversity. Numerical results are provided in Section IV to compare the proposed code with other replication-based literature coding schemes. Finally, Section V concludes this paper.

Notation: The notation used in this paper are as follows.

Boldface lower case letters $\mathbf{v}$ denote vectors, boldface capital letters $\mathbf{M}$ denote matrices. $\mathbf{M}^\top$ and $\mathbf{M}^\dagger$ denote respectively the transpose and the conjugate transpose of matrix $\mathbf{M}$. The Euclidean norm of a vector $\mathbf{v}$ is denoted $\|\mathbf{v}\|$ and $\mathbb{1}^{[n\times T]}$ represents the all-ones matrix. The matrix vectorization operation $\text{vec}(\mathbf{A})$ converts the $n \times m$ matrix into a $nm \times 1$ vector by consecutively stacking the columns of matrix $\mathbf{A}$ into $\text{vec}(\mathbf{A})$. Finally, $\mathbb{Z}$ and $\mathbb{Q}$ define respectively the set of integers and rational numbers.

II. FSO MIMO MODEL

In this section, we present the FSO optical $n_t \times n_r$ MIMO system model with single carrier IM/DD, the spatial and temporal channel variations and the optical channel characteristics.

A. FSO MIMO system

At a given receiver aperture i and instant t, the optical received power is converted to an electrical signal,

$$y_i(t) = r \sum_{j=1}^{n_t} h_{i,j}(t) x_j(t) + z_i(t), \qquad \forall 1 \le j \le n_r, \tag{1}$$

where r is the photo-diode responsivity and x_i is the input optical intensity transmitted by laser i. The transmitter front-end constrains the maximal and minimal intensity power $I_{\min} \le x_j(t) \le I_{\max}$ with $I_{\min}$ and $I_{\max}$ being the maximal and minimal intensity at each laser. Safety regulation imposes also an average transmit power $\mathbb{E}[x_j(t)] \le I_{\text{av}}$. In the rest of the paper, we consider without loss of generality that $I_{\max} = 2I_{\text{av}}$ and $I_{\min} = 0$. The optical channel gain $h_{i,j}(t)$ is induced by the deterministic path-loss, the atmospheric random scintillation and the misalignment loss. The additive noise $z_i(t)$ corresponds to the Gaussian thermal noise and the shot noise. Due to the strong ambient illumination, this latter is approximated by a Gaussian distribution and the total noise variance is denoted by σ_n^2. In the rest of the paper, the average Signal to Noise ratio is such that $\text{SNR} = (r^2\, \text{I}_{\text{av}}^2)/\sigma_n^2$.

B. Spatial and temporal variations

1) MIMO spatial variation: At a given instant t, the spatial channel coefficients $h_{i,j}(t)$ with $1 \le i \le n_r$ and $1 \le j \le n_t$, are considered as uncorrelated if the inter-spacing between lasers (respectively the apertures) is larger than the coherence length ℓ_c [1]. In the weak turbulence regime, $\ell_c \approx \sqrt{\lambda d}$. In the relatively strong regime with wave plane propagation, larger aperture spacing is required and $\ell_c \approx \lambda d/\rho_0(d)$ with $\rho_0(d) = (0.55\, C_{2,n}\, k^2\, d)^{-3/5}$ being the Fried coherence length, k the optical wave number and $C_{2,n}$ the index of refraction structure parameter [1].

2) Temporal variation: For FSO communications, the coherence time τ_c is in the order of 0.1 to 10 ms [1, 10]. The FSO channel remains then constant over a block of hundred of thousand bits for high transmission transmission data rates [1], [11]–[15] (that corresponds to τ_c) and it changes to a new independent value in the next block.

C. Statistical channel distribution

Each path-loss $h_{i,j}$ between the receiver aperture i and the laser j is computed as the product of the deterministic atmospheric path loss h_0, the random misalignment loss $h_{i,j}^{(p)}$ and the random atmospheric turbulence $h_{i,j}^{(a)}$ such that $h_{i,j} = h_0 h_{i,j}^{(a)} h_{i,j}^{(p)}$. The channel coefficients are assumed to be only known at the receiver side. The deterministic atmospheric path-loss is computed using the exponential Beers-Lambert Law $h_0 = e^{-\nu d}$ with d being the distance between the transmitter and the receiver and ν the attenuation factor that depends on the weather conditions [19].

1) Atmospheric turbulence distribution: The atmospheric turbulence distribution is modeled using the Gamma-Gamma distribution such that,

$$f(h_{i,j}^{(a)}) = \frac{2(\alpha\beta)^{(\alpha+\beta)/2}}{\Gamma(\alpha)\Gamma(\beta)} \big(h_{i,j}^{(a)}\big)^{(\alpha+\beta)/2-1} K_{\alpha-\beta}\left(2\sqrt{\alpha\beta h_{i,j}^{(a)}}\right),$$

where $K_{\alpha-\beta}$ is the modified Bessel function of the second kind, $\Gamma(.)$ is the Gamma function. The parameters α and $\beta < \alpha$ represent large and small-scale eddies of the scattering environment and are related to the Rytov variance σ_R^2 defined as $\sigma_R^2 = 1.23\, C_{2,n}\, k^{7/6}\, d^{11/6}$ with parameters α and β obtained from [21],

$$\begin{aligned} \alpha^{-1} &= \exp\left(\frac{0.49\sigma_R^2}{(1+1.11\sigma_R^{12/5})^{7/6}}\right) - 1, \\ \beta^{-1} &= \exp\left(\frac{0.51\sigma_R^2}{(1+0.69\sigma_R^{12/5})^{5/6}}\right) - 1. \end{aligned}$$

2) Misalignment distribution: To model the impact of pointing errors, we use the misalignment statistical model given in [19, 20] as $h_{i,j}^{(p)} \approx A_0\ e^{-2r_{i,j}^2/w_{\text{eq}}^2}$ where A_0 is the equivalent receiver area, w_{eq} is the equivalent beam waist at the receive and $r_{i,j}$ is the radial displacement between the center of the transmitter beam j and the center of the aperture i. The radial displacement $r_{i,j}$ is modeled by a Rician distribution parametrized by the jitter standard deviation σ_s and the non-zero boresight displacement s, as in [20], such that

$$f(r_{i,j}) = \frac{r_{i,j}}{\sigma_s^2} \exp\left(-\frac{(r_{i,j}+\sigma_s)^2}{2\sigma_s^2}\right) I_0\left(\frac{r_{i,j}\, s}{\sigma_s^2}\right),$$

with $I_0(.)$ being the modified Bessel function of the first kind with order zero. The values of A_0 and w_{eq} are detailed in [Eq (9) in [19]].

III. EXTENDED GOLDEN LIGHT CODE (E-GLC) CONSTRUCTION

In this section, we focus on block fading MIMO systems with $n_t = 2$ transmitting lasers, $n_r \ge 2$ receiver aperture and two fading blocks. Each packet of ℓ bits is encoded in two different ways using optical Space Time Coding (STC) schemes matrices $\mathbf{X}_{o,1}$ and $\mathbf{X}_{o,2}$ transmitted each on two independent blocks. The delay between $\mathbf{X}_{o,1}$ and $\mathbf{X}_{o,2}$ should be at least equal to the coherence time τ_c.

A. Optical constraints

In order to constitute the optical STC schemes $\mathbf{X}_{o,1}$ and $\mathbf{X}_{o,2}$, the stream of ℓ bits is first converted into symmetric 2^q-PAM symbols with alphabet defined as,

$$\mathcal{A}_q = \{\pm(2^q - 2n + 1), \qquad 1 \leq n \leq 2^{q-1}\}.$$

The main objective is to construct a full rate family of code $\mathcal{X}_c$ with symbol rate of $r = 2$ that extracts the temporal and the spatial diversity. To extract the spatial diversity and the multiplexing gains, the optical code spreads, on each block, the combined symbols on $T = 2$ time slots. Over the two blocks the total number of symbols is $2rT = 8$ symbols. Each PAM symbol carries q bits per channel use (bpcu) and the packet length is then $\ell = 8q$. The size of constellation is determined with respect to the target spectral efficiency R bpcu. The full symbol rate being equal to 2, then $R = 2q$ (bpcu). To meet the peak and average intensity constraints at each transmitting laser, we build the optical STC schemes $\mathbf{X}_{o,1}$ and $\mathbf{X}_{o,2}$ on matrices $\mathbf{X}_{c,1}$ and $\mathbf{X}_{c,2}$ with real and zero-mean components each (having a maximal and minimal component values $x_{c,\max}$ and $x_{c,\min}$ satisfying $x_{c,\max} = -x_{c,\min}$. The components of $\mathbf{X}_{c,1}$ and $\mathbf{X}_{c,2}$ are computed by adequately combining symbols belonging to a symmetric 2^q-PAM constellation. Finally, a bipolar to unipolar conversion is performed by scaling the zero-mean components matrix and adding a DC bias such that

$$\mathbf{X}_{o,k}^{[2\times 2]} = \varepsilon\, \mathbf{X}_{\mathrm{c},k}^{[2\times 2]} + \mathrm{I}_{\mathrm{av}}\, \mathbb{1}^{[2\times 2]}, \qquad k = 1, 2, \tag{2}$$

with,

$$\varepsilon = \frac{\mathrm{I}_{\mathrm{av}}}{2^q - 1} \varepsilon_c$$

being the scaling factor that guarantees that the peak intensity constraint is satisfied, and

$$\varepsilon_c = (2^q - 1)\, x_{c,\max}^{-1}$$

the code scaling factor. The received signal in (1) can be rewritten as an equivalent MIMO block fading such that,

$$\begin{bmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{bmatrix} = \frac{r\,\mathrm{I}_{\mathrm{av}}}{2^q - 1}\varepsilon_c \begin{bmatrix} \mathbf{H}_1 & 0 \\ 0 & \mathbf{H}_2 \end{bmatrix} \begin{bmatrix} \mathbf{X}_{c,1} \\ \mathbf{X}_{c,2} \end{bmatrix} + \begin{bmatrix} \mathbf{Z}_1 \\ \mathbf{Z}_2 \end{bmatrix} \tag{3}$$

where $\mathbf{Y}_1, \mathbf{Y}_2 \in \mathbb{R}^{[n_r \times 2]}$ are the received signal matrices, $\mathbf{H}_1, \mathbf{H}_2 \in \mathbb{R}^{+,[n_r \times n_t]}$ are the uncorrelated optical channels, and $\mathbf{Z}_1, \mathbf{Z}_2 \in \mathbb{R}^{[n_r \times 2]}$ are the Gaussian noise matrices.

B. Code design criteria

The approximately universal criterion in [22]–[24] applied to the block diagonal model in (3) with a target spectral efficiency and considering real data input is such that,

$$\min_{\mathbf{X}_c \in \mathcal{X}_c} \prod_{i=1}^{2} \det\left(\frac{1}{(2^q - 1)^2}\, \varepsilon_c^2\, \Delta\mathbf{X}_{c,i} \Delta\mathbf{X}_{c,i}^{\top} \right) \dot{\geq} 2^{-4R}. \tag{4}$$

Note here that we used 2^{-4R} instead of 2^{-2R} in [22]–[24] as we are dealing with real inputs and not with complex ones as in [22]–[24] which will induce a factor of $1/2$ in the capacity expression. Moreover, the scaling factor ε_c depends on the structure of the code and should be considered in the determinant maximization. Using a full-rate symbol rate code with symbol rate of 2, then $R = 2q$ bpcu and $(2^q - 1)^2 \doteq 2^R$ then the approximately universal code is equivalent to the Non-Vanishing Determinant (NVD) criteria,

$$\delta_{\mathrm{min},c}^2 \triangleq \min_{\mathbf{X}_c \in \mathcal{X}_c} \prod_{i=1}^{2} \det\left(\varepsilon_c^2\, \Delta\mathbf{X}_{c,i} \Delta\mathbf{X}_{c,i}^{\top} \right) \dot{\geq} 1. \tag{5}$$

The approximately universal code design guarantees that the error probability has the same SNR exponent as the outage probability independently of the channel distribution [22]–[24]. For a fixed spectral efficiency that does not grow in function of SNR, the outage SNR exponent is nothing but the full diversity order that can be achieved. Our main objective is to define matrices $\mathbf{X}_{c,1}$ and $\mathbf{X}_{c,2}$ that verify the NVD criteria for which $\delta_{\mathrm{min},c}^2$ is maximized independently of the channel distribution. Defining such type of code guarantees that the full temporal and spatial diversity order is extracted.

C. Extended Golden Light Code (E-GLC) code structure

The matrices components of the proposed E-GLC scheme $\mathbf{X}_{c,1}$ and $\mathbf{X}_{c,2}$ are constructed by combining eight symbols belonging to a 2^q-PAM $s_1, s_2, s_3, s_4, s_5, s_6, s_7$ and s_8 using orthonormal matrix parametrized by z_1, z_2, z_3 and z_4, and their respective reciprocals $\bar{z}_1 = z_1^{-1}$, $\bar{z}_2 = z_2^{-1}$, $\bar{z}_3 = z_3^{-1}$ and $\bar{z}_4 = z_4^{-1}$, such that:

$$\mathbf{X}_{c,1}^{[2\times 2]} = \begin{bmatrix} x_{1,1}^{(1)} & x_{1,2}^{(1)} \\ x_{2,1}^{(1)} & x_{2,2}^{(1)} \end{bmatrix}, \tag{6}$$

where,

$$\begin{aligned}
x_{1,1}^{(1)} &= \frac{1}{\sqrt{(1+z_1^2)(1+z_2^2)}} \big[(s_1 - z_2 s_2) - z_1(s_3 - z_2 s_4)\big], \\
x_{2,1}^{(1)} &= \frac{1}{\sqrt{(1+z_3^2)(1+z_4^2)}} \big[(s_5 - z_4 s_6) - z_3(s_7 - z_4 s_8)\big], \\
x_{1,2}^{(1)} &= \frac{z_3}{\sqrt{(1+z_3^2)(1+z_4^2)}} \big[(s_5 - z_4 s_6) + \bar{z}_3(s_7 - z_4 s_8)\big], \\
x_{2,2}^{(1)} &= \frac{z_1}{\sqrt{(1+z_1^2)(1+z_2^2)}} \big[(s_1 - z_2 s_2) + \bar{z}_1(s_3 - z_2 s_4)\big],
\end{aligned}$$

and

$$\mathbf{X}_{c,2}^{[2\times 2]} = \begin{bmatrix} x_{1,1}^{(2)} & x_{1,2}^{(2)} \\ x_{2,1}^{(2)} & x_{2,2}^{(2)} \end{bmatrix}, \tag{7}$$

where,

$$\begin{aligned}
x_{1,1}^{(2)} &= \frac{z_2}{\sqrt{(1+z_1^2)(1+z_2^2)}} \big[(s_1 + \bar{z}_2 s_2) - z_1(s_3 - \bar{z}_2 s_4)\big], \\
x_{2,1}^{(2)} &= \frac{z_4}{\sqrt{(1+z_3^2)(1+z_4^2)}} \big[(s_5 + \bar{z}_4 s_6) - z_3(s_7 + \bar{z}_4 s_8)\big], \\
x_{1,2}^{(2)} &= \frac{z_4 z_3}{\sqrt{(1+z_3^2)(1+z_4^2)}} \big[(s_5 + \bar{z}_4 s_6) + \bar{z}_3(s_7 + \bar{z}_4 s_8)\big], \\
x_{2,2}^{(2)} &= \frac{z_1 z_2}{\sqrt{(1+z_1^2)(1+z_2^2)}} \big[(s_1 + \bar{z}_2 s_2) + \bar{z}_1(s_3 + \bar{z}_2 s_4)\big].
\end{aligned}$$

The bipolar to unipolar scaling factor is,

$$\varepsilon_c(z_1, z_2, z_3, z_4) = \min\left(\frac{\sqrt{(1+z_1^2)(1+z_2^2)}}{(1+|z_1|)(1+|z_2|)}\,; \right.$$

$$\left. \frac{\sqrt{(1+z_3^2)(1+z_4^2)}}{(1+|z_3|)(1+|z_4|)}\right). \quad (8)$$

The main objective of the paper is then to define a family of code $\mathcal{X}_c$ that maximizes the minimal determinant $\delta_{\min,c}$ that depends on z_1, z_2, z_3 and z_4 such that,

$$\delta_{\min,c}(z_1, z_2, z_3, z_4) = \max_{z_1,z_2,z_3,z_4} \min_{\Delta s_i \in \mathcal{R}_q} \varepsilon_c^4 \det(\Delta \mathbf{X}_1) \det(\Delta \mathbf{X}_2),$$

with

$$\mathcal{R}_q = \left\{ \Delta s_{1_8} \in \mathbb{Z}^{*8} : |\Delta s_1|, \dots, |\Delta s_8| \leq 2^q - 1 \right\}. \quad (9)$$

The main result of this paper is summarized in the following theorem:

Theorem 1. *For $\Delta s_i \in \mathbb{Z}$ with $1 \leq i \leq 8$, the values of z_1, z_2, z_3 and z_4 that maximize the function $\delta^2_{\min,d}(z_1, z_2, z_3, z_4)$ are such that $z_1 - \bar{z}_1 = \pm 1$, $z_2 - \bar{z}_2 = \pm 5$, $z_3 - \bar{z}_3 = \pm 2$ and $z_4 - \bar{z}_4 = \pm 5$, or,*

$$\begin{aligned} z_1^* &= \frac{1}{2}\big(\pm 1 \pm \sqrt{5}\big), \\ z_2^* &= \frac{1}{2}\big(\pm 5 \pm \sqrt{29}\big), \\ z_3^* &= \pm 2 \pm \sqrt{5}, \\ z_4^* &= \frac{1}{2}\big(\pm 5 \pm \sqrt{29}\big). \end{aligned}$$

Indeed, $\delta_{\min,c}(z_1^*, z_2^*, z_3^*, z_4^*)$ is only and only equal to zero for the trivial solution $\Delta s_1 = \Delta s_2 = \ldots = \Delta s_8 = 0$ and its minimal value is

$$\delta^*_{\min,c} = \varepsilon_c^{*4} \frac{z_2^{*2} z_3^{*2}}{(1+z_2^{*2})^2 (1+z_3^{*2})^2}$$

with

$$\varepsilon_c^* = \frac{\sqrt{(1+z_1^{*2})(1+z_2^{*2})}}{(1+|z_1^*|)(1+|z_2^*|)}.$$

Proof: The proof of this lemma is detailed in Appendix A. ∎

We can notice that the E-GLC is nothing but the extension of the Golden Light Code (GLC) structure proposed in [3, 4] that encodes 4 symbols a_1, a_2, a_3, a_4 belonging to a 2^q-PAM constellation (a subset of $\mathbb{Z}$) into a one-block matrix $\mathbf{X}_c$ defined in [3, 4] as,

$$\mathbf{X}_c = \begin{bmatrix} \frac{1}{\sqrt{1+z_1^*}}(a_1 + z_1^* a_2) & \frac{1}{\sqrt{1+z_3^*}}(a_3 + z_3^* a_4) \\ \frac{z_3^*}{\sqrt{1+z_3^*}}(a_3 + \bar{z}_3^* a_4) & \frac{z_1^*}{\sqrt{1+z_1^*}}(a_1 + \bar{z}_1^* a_2) \end{bmatrix}. \quad (10)$$

For the E-GLC, the first block $\mathbf{X}_{c,1}$ is constituted by extending the space of a_i inputs in (10) (which is $\mathbb{Z}$ in the GLC case) to a higher dimensional ring of integer space $\mathcal{O}_{\mathbb{K}_2}$ with minimal polynomial $X^2 - 5X - 1 = 0$ having roots z_2 and $-\bar{z}_2 = -z_2^{-1}$, such that,

$$a_i = \frac{1}{\sqrt{1+z_2^2}}(s_{2i-1} - z_2 s_{2i}), \qquad 1 \leq i \leq 4,$$

with $(s_{2i-1} - z_2 s_{2i}) \in \mathcal{O}_{\mathbb{K}_2}$ and $(s_{2i-1} + \bar{z}_2 s_{2i})$ being its conjugate. The second E-GLC block $\mathbf{X}_{c,2}$ is constituted by replacing the inputs a_i in (10) by the normalized conjugate of the algebraic integers such that,

$$\bar{a}_i = \frac{z_2}{\sqrt{1+z_2^2}}(s_{2i-1} + \bar{z}_2 s_{2i}), \qquad 1 \leq i \leq 4.$$

IV. ALTERNATIVE SCHEMES AND NUMERICAL RESULTS

In this section, we propose to evaluate the performance of E-GLC scheme in terms of Packet Error Rate (PER) versus the average electrical SNR $= (r^2\, \mathrm{I}_{\mathrm{av}}^2)/\sigma_n^2$ and to compare it to other literature schemes. To make a fair comparison between the performances of different STCs, we fix the same bandwidth, average and maximal transmit power and we target the same spectral efficiency R bits per channel use (bpcu). The simulation parameters of the FSO system are summarized in Table I. The value of the refractive index $C_{2,n}$ varies from 10^{-13} m$^{-2/3}$ for strong turbulence to 10^{-17} m$^{-2/3}$ for weak turbulence with a typical average value in order of 10^{-15} m$^{-2/3}$. We choose, without loss of generality, the moderate turbulence characterized by $C_{2,n} = 5 \times 10^{-15}$ m$^{-2/3}$. The coherence length that guarantees spatial independence in the order of 20 cm. In the moderate turbulence case, a typical coherence time value is $\tau_c = 100\mu$s. We set the data rate to $D_b = 200$ Mbps with a bit duration $T_b = 5$ ns. To extract the temporal diversity, the burst duration should be higher than $2\tau_c = 200\mu$s. The bandwidth is determined in function of the target spectral efficiency R as $B = D_b/R$ and the symbol slot duration is fixed to $T_s = R\ T_b$. For all the considered schemes, we assume that the symbol pulses have the same width T_s and that the same bandwidth B is allocated.

TABLE I
OPTICAL SYSTEM PARAMETERS

Parameter	Value
Turbulence strength	Moderate
Refractive index $C_{2,n}$	5×10^{-15} m$^{-2/3}$
Attenuation factor ν	7.82 km^{-1}
Spatial coherence length	20 cm
Coherence time	100 μs
Link distance d	5 km
Data rate	200 Mbps
Burst duration	200 μs
Delay latency	200 μs
Lasers/apertures spacing	25 cm
Wavelength λ	1550 nm
Receiver radius a	10 cm
Beam waist radius w_d	2.5 m
Typical misalignment variance σ_s	30 cm

A. Encoder and decoder of the E-GLC

The E-GLC combines 8 symbols $s_1, s_2, \dots, s_8$ and encodes them in different ways over the two fading blocks according to $\mathbf{X}_{c,1}$ and $\mathbf{X}_{c,2}$ in (6) and (7). These two blocks are separated by a time duration larger than τ_c that guarantees uncorrelated fading channels. The matrices components are generated using the orthonormal matrices $\mathbf{R}_1$ and $\mathbf{R}_2$ for which the numerical values computed in Theorem 1 are such that,

$$\mathbf{R}_1 = \begin{bmatrix} 0.0994 & -0.5162 & -0.1609 & 0.8353 \\ 0.1609 & -0.8353 & 0.0994 & -0.5162 \\ 0.5162 & 0.0994 & -0.8353 & -0.1609 \\ 0.8353 & 0.1609 & 0.5162 & 0.0994 \end{bmatrix}$$

and

$$\mathbf{R}_2 = \begin{bmatrix} 0.0434 & -0.2256 & -0.184 & 0.9557 \\ 0.184 & -0.9557 & 0.0434 & -0.2256 \\ 0.2256 & 0.0434 & -0.9557 & -0.184 \\ 0.9557 & 0.184 & 0.2256 & 0.0434 \end{bmatrix}$$

such that,

$$\begin{bmatrix} x_{1,1}^{(1)} \\ x_{2,2}^{(1)} \\ x_{1,1}^{(2)} \\ x_{2,2}^{(2)} \end{bmatrix} = \mathbf{R}_1 \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{bmatrix}, \quad \begin{bmatrix} x_{2,1}^{(1)} \\ x_{1,2}^{(1)} \\ x_{2,1}^{(2)} \\ x_{1,2}^{(2)} \end{bmatrix} = \mathbf{R}_2 \begin{bmatrix} s_5 \\ s_6 \\ s_7 \\ s_8 \end{bmatrix};$$

Let $\mathbf{x}$ (resp. $\mathbf{y}$ and $\mathbf{z}$) denote the vectorized transmitted signal (resp. the received signal and the additive noise) such that,

$$\mathbf{x} = \mathrm{vec}\left(\begin{bmatrix} \mathbf{X}_{c,1} \\ \mathbf{X}_{c,2} \end{bmatrix} \right)$$

and

$$\mathbf{y} = \mathrm{vec}\left(\begin{bmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{bmatrix} \right), \quad \mathbf{z} = \mathrm{vec}\left(\begin{bmatrix} \mathbf{Z}_1 \\ \mathbf{Z}_2 \end{bmatrix} \right).$$

The global code rotation matrix denoted by $\mathbf{R}$ can be deduced by rearranging the rows of $\mathbf{R}_1$ and $\mathbf{R}_2$ to write $\mathbf{x} = \mathbf{R}\mathbf{s}$ with $\mathbf{s} = [s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8]^\top$. Using the E-GLC code, the block fading model in (3) is equivalent to,

$$\mathbf{y} = \frac{r\,\mathrm{I}_{\mathrm{av}}}{2^q - 1} \varepsilon_c\, \mathbf{M}\mathbf{s} + \mathbf{z}$$

with $\mathbf{M}$ is the equivalent lattice-generating matrix given by,

$$\mathbf{M} = \begin{bmatrix} \mathbf{H} & \mathbf{0} \\ \mathbf{0} & \mathbf{H} \end{bmatrix} \mathbf{R},$$

and $\mathbf{H} = \mathrm{diag}\{\mathbf{H}_1, \mathbf{H}_2\}$. A the receiver side, a Maximum-Likelihood (ML) based on the sphere decoder algorithm is performed to jointly decode the eight symbols by finding the vector $\hat{\mathbf{s}}$ that minimizes the Euclidean distance,

$$\hat{\mathbf{s}} = \arg \min_{\mathbf{s} \in \mathcal{A}_q^8} \left\| \mathbf{y} - \frac{r\,\mathrm{I}_{\mathrm{av}}}{2^q - 1} \varepsilon_c\, \mathbf{M}\mathbf{s} \right\|^2.$$

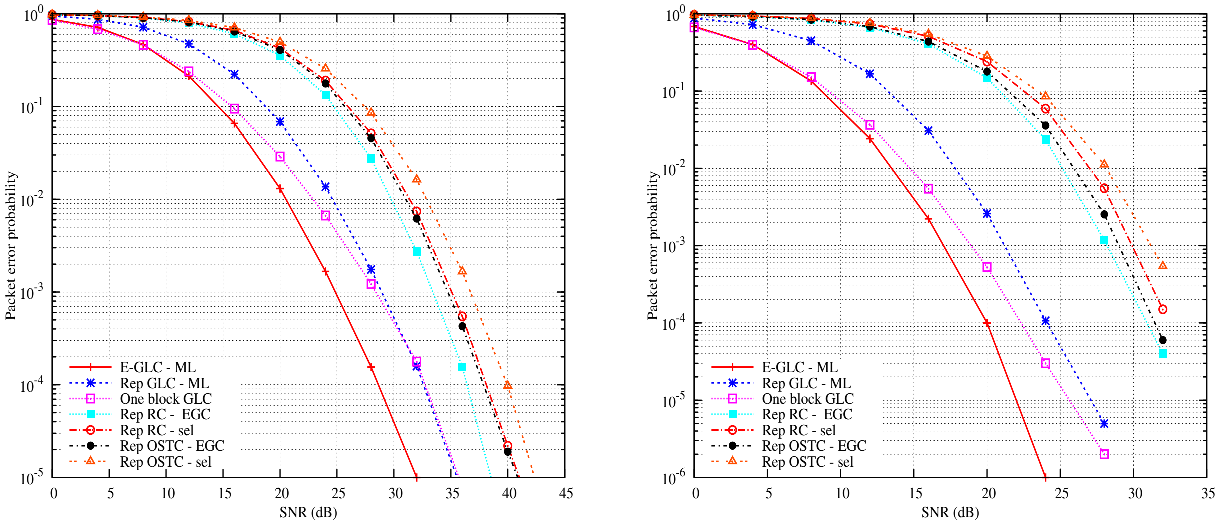
Finally, the code scaling factor that adjusts the signal to the peak intensity constraint is $\varepsilon_c = 0.6204$ which is lower than the GLC scaling factor of 0.7265 computed in [3, 4].

B. Literature coding schemes: signal replication

Most of the literature schemes that extract the spatial and temporal diversity are based on a replication of the same one-block STC $\mathbf{X}_{c,1} = \mathbf{X}_{c,2}$ over two i.i.d. blocks. As in the E-GLC case, the decoding cannot be performed before the reception of the two signal replications, *i.e.* after a delay of at least τ_c. Although this signal replication extracts the temporal and spatial diversity, this comes at the expense of a decreased symbol rate. To extract the spatial diversity, the one-block STC can be either a Repetition Code (RC), an Orthogonal Space Time Code (OSTC) or a Golden Light Code (GC) as [3]–[8] . In all cases, the bits are mapped into PAM symbols with a constellation size that depends on the target spectral efficiency. Then, the matrices $\mathbf{X}_{c,1} = \mathbf{X}_{c,2}$ are then constructed accordingly. The RC simply repeats the same symbol over all the lasers. The OSTC uses an Alamouti Code as in [5]. Using a replicated-RC or a replicated-OSTC Alamouti code results in a symbol rate of only $1/2$. For the RC and the OSTC Alamouti code, different signal decoding techniques were adopted. The Equal Gain Combination (EGC) at the receiver combines all signal contributions over the two fading blocks. Another alternative is to select the best received signal among the two blocks as in [15]. A replicated coding scheme can be constructed as well using the Golden Light Code (GLC) proposed in [3, 4]. Over one block, this code carries 4 symbols during two time slots; the same code is replicated over the next block. The symbol rate is in this case equal to 1. At the receiver side, the temporal diversity of the Replicated-GLC code can be extracted using a ML decoder applied over the $2n_r \times 2$ stack matrix $\left[\mathbf{H}_1^\top \ \mathbf{H}_2^\top\right]^\top$. The GLC is an NVD code and it extracts the full diversity in both configuration with only spatial diversity (over a $n_r \times 2$ MIMO channel) and with temporal and spatial diversity (over the $2n_r \times 2$ stack matrix).

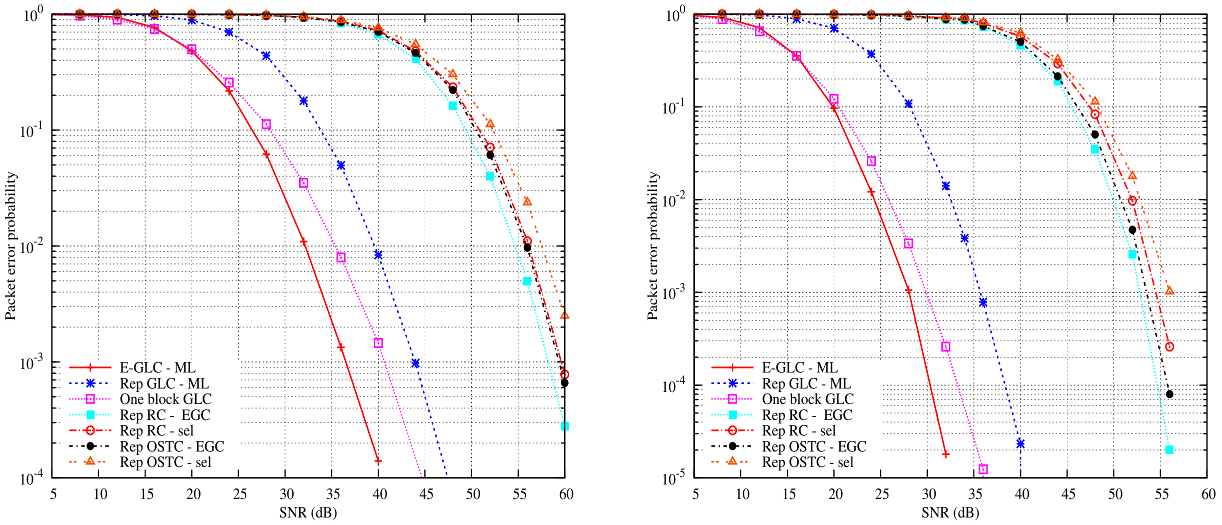
C. Error rate comparison

Figure 1 compares the PER of these schemes for 2×2 and 2×4 MIMO configurations considering two independent i.i.d. fading blocks with target rates of $R = 2$ bpcu and $R = 4$ bpcu. The E-GLC is a full rate code with symbol rate of 2 that extracts the spatial and the temporal diversity. In order to achieve the target rate of 2 bpcu (resp. 4 bpcu), a BPSK (resp. 4-PAM) constellation is required. Replicating the one-block over two blocks reduces in half the symbol rate of this STC code. To compensate this loss in symbol rate, higher constellation size should be used. The size of the PAM constellation should be then adjusted to compensate the loss in symbol rate. The replicated RC and replicated OSTBC Alamouti code with a symbol rate of $1/2$ require 16-PAM (resp. 256-PAM) constellation to achieve R = 2 bpcu in Figures (resp. 4 bpcu). The replicated GLC with symbol rate of 1 requires a 4-PAM (resp. 16-PAM) constellation is required to achieve these same target efficiencies. Considering only the spatial diversity, distinct GLC coding schemes can be used over the independent blocks. There is no loss in the GLC symbol rate that remains equal to 2, however the temporal diversity is not extracted. For a 2×2 MIMO configuration with different spectral efficiencies R = 2 bpcu (resp. R = 4 bpcu) in Figures 1(a) and Figure 1(c), we can see that the full symbol rate and the full spatial and temporal diversity of the E-GLC code enhance significantly its PER performance with other space time coding schemes. The one-block GLC conserves the full rate but it does not extract the time diversity as it can be seen from the slope of its PER curve. The low symbol rate of the RC and OSTBC penalize significantly their



(a) 2×2 MIMO - Target rate of $R = 2$ bpcu

(b) 2×4 MIMO - Target rate of $R = 2$ bpcu



(c) 2×2 MIMO - Target rate of $R = 4$ bpcu

(d) 2×4 MIMO - Target rate of $R = 4$ bpcu

Fig. 1. PER of 2×2 and 2×4 MIMO configuration considering target rates $R = 4$ bpcu

performance as higher constellation are required to compensate this loss in symbol rate. Similar observations are observed when considering higher order of receiver diversity using a 2×4 MIMO configuration in Figures 1(b) and Figure 1(d). This additional diversity enhance the error performance of all STC schemes compared to 2×2 MIMO case.

V. CONCLUSION

In this paper, we proposed an Extended Golden Light Code (E-GLC) construction with Non Vanishing Determinant (NVD) and symbol rate of two for optical transmission over two independent fading blocks of a $2 \times n_r$ MIMO FSO channel with IM/DD. The NVD code construction guarantees that the full diversity order is achieved independently of the channel statistical distribution. The proposed code construction is based on an adequate choice of quadratic fields that extend the $\mathbb{Z}$ space into higher dimensional space. These algebraic numbers combine the PAM symbols in such a way that the code determinant is non-vanishing and the code scaling factor is maximized. We show that this code construction extends the Golden Light Code construction in [3, 4] by considering a higher dimensional space of inputs symbols. Compared to replication-based schemes, we show that the coding gain induced by the full symbol rate of this E-GLC enhances significantly the error performance.

APPENDIX A
PROOF OF THEOREM 1

In $\mathcal{R}_1$, the optimal solution that maximizes $\delta_{\min,c}$ is obtained for $z_1^* - \bar{z}_1^* = \pm 1$, $z_2^* - \bar{z}_2^* = \pm 5$, $z_3^* - \bar{z}_3^* = \pm 2$ and $z_4^* - \bar{z}_4^* = \pm 5$. For the general 2^q-PAM case, we show that this solution is also optimal. Considering the general set $\mathcal{R}_q$ case in (9), we can write the determinant of $\mathbf{X}_{c,1}$ and $\mathbf{X}_{c,2}$ in

$$\begin{aligned} N_{\mathbb{F}/\mathbb{K}_2}[x] &= a(z_2^*)^2 - b(z_2^*)^2 - a(z_2^*)b(z_2^*) \\ &= (a_1^2 + a_2^2 z_2^{*2} - 2a_1a_2z_2^*) - (b_1^2 + b_2^2 z_2^{*2} - 2b_1b_2z_2^*) - (a_1b_1 - z_2^*a_1b_2 - z_2^*a_2b_1 + z_2^{*2}a_2b_2) \\ &= N_{\mathbb{K}_1/\mathbb{Q}}[a_1 - z_1^*b_2] - z_2^*\,\mathrm{Tr}_{\mathbb{K}_1/\mathbb{Q}}[(a_1 - z_1^*b_1)(a_2 + \bar{z}_1^*b_2)] + z_2^{*2}N_{\mathbb{K}_1/\mathbb{Q}}[a_2 - z_1^*b_2] \\ &= N_{\mathbb{K}_1/\mathbb{Q}}[c(z_1^*)] + N_{\mathbb{K}_1/\mathbb{Q}}[d(z_1^*)] - z_2^*\left\{\mathrm{Tr}_{\mathbb{K}_1/\mathbb{Q}}[c(z_1^*)\bar{d}(z_1^*)] - 5N_{\mathbb{K}_1/\mathbb{Q}}[d(z_1^*)]\right\} \qquad (11) \end{aligned}$$

$$\begin{aligned} \mathrm{Tr}_{\mathbb{K}_1/\mathbb{Q}}[c(z_1^*)\bar{d}(z_1^*)] &= a_1a_2 + \bar{z}_1^*a_1b_2 - z_1^*b_1a_2 - b_1b_2 + a_1a_2 - z_1^*a_1b_2 + \bar{z}_1^*b_1a_2 - b_1b_2 \\ &= 2(a_1a_2 - b_1b_2) - (a_2b_1 + a_1b_2) \qquad (12) \end{aligned}$$

---

(6) and (7) as,

$$\begin{aligned} \det(\Delta\bar{\mathbf{X}}_{c,1}) &= \frac{1}{1+z_2^{*2}}\frac{z_3^*}{1+z_3^{*2}}\,h_1(\Delta s_{1:8}, z_{1:4}^*) \\ \det(\Delta\bar{\mathbf{X}}_{c,2}) &= \frac{z_2^{*2}}{1+z_2^{*2}}\frac{z_3^*}{1+z_3^{*2}}\,h_2(\Delta s_{1:8}, z_{1:4}^*). \end{aligned}$$

with

$$\begin{aligned} h_1(\Delta s_{1:8}, z_{1:4}^*) = {}& 2(x_1 - x_2z_1^*)(x_1 - x_2\bar{z}_1^*) \\ & - \big((x_3 - x_4) - 2x_4z_1^*\big)\big((x_3 - x_4) - 2x_4\bar{z}_1^*\big) \quad (13) \end{aligned}$$

and

$$\begin{aligned} h_2(\Delta s_{1:8}, z_{1:4}^*) = {}& 2(\bar{x}_1 - \bar{x}_2z_1^*)(\bar{x}_1 - \bar{x}_2\bar{z}_1^*) \\ & - \big((\bar{x}_3 - \bar{x}_4) - 2\bar{x}_4z_1^*\big)\big((\bar{x}_3 - \bar{x}_4) - 2\bar{x}_4\bar{z}_1^*\big), \quad (14) \end{aligned}$$

such that $x_1 = \Delta s_1 - z_2^*\Delta s_2$ $x_2 = \Delta s_3 - z_2^*\Delta s_4$, $x_3 = \Delta s_5 - z_2^*\Delta s_6$, $x_4 = \Delta s_7 - z_2^*\Delta s_8$, $\bar{x}_1 = \Delta s_1 + \bar{z}_2\Delta s_2$ $\bar{x}_2 = \Delta s_3 + \bar{z}_2\Delta s_4$, $x_3 = \Delta s_5 + \bar{z}_2\Delta s_6$ and $\bar{x}_4 = \Delta s_7 + \bar{z}_2\Delta s_8$. We need to prove that $h_1(.)$ and $h_2(.)$ are only and only equal to zero for the trivial solution. Using (6) and (7), $h_1(.)$ and $h_2(.)$ can be written in function of the relative norm of the field $\mathbb{F}/\mathbb{K}_2$ defined as

$$\mathbb{F} = \mathbb{Q}(z_1^*, z_2^*) = \mathbb{K}_2[z_1^*] = \{a + bz_1^* | a, b \in \mathbb{K}_2\}$$

with

$$\mathbb{K}_2 = \mathbb{Q}[z_2^*] = \{c + dz_2^* | c, d \in \mathbb{Q}\}$$

the quadratic extension of $\mathbb{Q}$ and having $\mathcal{O}_{\mathbb{K}_2}$ as ring of algebraic integer such that,

$$\mathcal{O}_{\mathbb{K}_2} = \mathbb{Z}[z_2^*] = \{c + dz_2^* | a, b \in \mathbb{Z}\}$$

For a given algebraic number $a + bz_1^* \in \mathbb{F}$, the relative norm of $\mathbb{F}$ is

$$N_{\mathbb{F}/\mathbb{K}_2}[a + bz_1^*] = (a + bz_1^*)(a - b\bar{z}_1^*) \in \mathbb{K}_2$$

with $a, b \in \mathbb{K}_2$. The expressions of $h_1(.)$ and $h_2(.)$ are then,

$$\begin{aligned} h_1(.) &= 2N_{\mathbb{F}/\mathbb{K}_2}\left[x_1 - x_2z_1^*\right] - N_{\mathbb{F}/\mathbb{K}_2}[(x_3 - x_4) - 2x_4z_1^*], \\ h_2(.) &= 2N_{\mathbb{F}/\mathbb{K}_2}\left[\bar{x}_1 - \bar{x}_2z_1^*\right] - N_{\mathbb{F}/\mathbb{K}_2}[(\bar{x}_3 - \bar{x}_4) - 2\bar{x}_4z_1^*], \end{aligned}$$

where $x_1, x_2, x_3, x_4 \in \mathcal{O}_{\mathbb{K}_2}$ and $\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4$ are their respective conjugates in $\mathcal{O}_{\mathbb{K}_2}$. We can notice that $h_2(.)$ is the conjugate of $h_1(.)$ in the ring of integer $\mathcal{O}_{\mathbb{K}_2}$. Therefore, $h_1(.)h_2(.)$ is the relative norm of $\mathbb{K}_2$ in $\mathbb{Z}$. The non-zero minimal absolute value of $|h_1(.)h_2(.)|$ in $\mathbb{Z}$ is 1. We need to show that $h_1 \in \mathbb{O}_{\mathbb{K}_2}$ (or equivalently its conjugate $h_2(.)$) cannot be equal to zero except for the trivial solution.

If $\Delta s_1 = \ldots = \Delta s_8 = 0$, then $x_1 = x_2 = x_3 = x_4 = 0$, and $h_1(.) = h_2(.) = 0$. Conversely, suppose that $(x_1, x_2, x_3, x_4) \neq (0, 0, 0, 0)$, we need to show that it is impossible to find $(x_1, x_2, x_3, x_4) \neq (0, 0, 0, 0)$ for which $h_1(.) = 0$. Assume that

$$h_1(.) = 0 \Leftrightarrow 2N_{\mathbb{F}/\mathbb{K}_2}[x] = N_{\mathbb{F}/\mathbb{K}_2}[y]$$

with $x = (x_1 - x_2z_1^*) \in \mathbb{F}$ and $y = (x_3 - x_4) - 2x_4z_1^* \in \mathbb{F}$. Knowing that $N_{\mathbb{F}/\mathbb{K}_2}[x]$ and $N_{\mathbb{F}/\mathbb{K}_2}[y]$ are $\in \mathbb{K}_2$ and after checking in Sage [25] that 2 is prime in $\mathbb{K}_2$, we can state that,

$$h_1(.) = 0 \Leftrightarrow (N_{\mathbb{F}/\mathbb{K}_2}[y] = 2) \text{ or } (N_{\mathbb{F}/\mathbb{K}_2}[x] = N_{\mathbb{F}/\mathbb{K}_2}[y] = 0)$$

This requires in turn the following lemma:

Lemma 1. *The two following statements are verified in* $\mathbb{F}$*:*

1) $\forall x \in \mathbb{F} = \mathbb{Q}(z_1^*, z_2^*)$*, we have* $N_{\mathbb{F}/\mathbb{K}_2}[x] \neq 2$*;*
2) $N_{\mathbb{F}/\mathbb{K}_2}[x] = 0 \Leftrightarrow x = 0$.

Proof: The proof of this lemma is detailed in Appendix B. ■

Lemma 1 implies it is impossible to find $(x_1, x_2, x_3, x_4) \neq (0, 0, 0, 0)$ for which $h_1(.) = 0$. Since $h_1(.)h_2(.)$ is in $\mathbb{Z}$, the minimal non-zero value is therefore ±1, and thus,

$$\delta_{\min,c} = \varepsilon_c^{*4}\frac{z_2^{*2}z_3^{*2}}{(1+z_2^{*2})^2(1+z_3^{*2})^2}$$

with

$$\varepsilon_c^* = \frac{\sqrt{(1+z_1^{*2})(1+z_2^{*2})}}{(1+|z_1^*|)(1+|z_2^*|)}.$$

APPENDIX B
PROOF OF LEMMA 1

Let x be a non-zero algebraic number $\in \mathbb{F}$ defined as:

$$\begin{aligned} x &= (a_1 - a_2z_2^*) - (b_1 - z_2^*b_2)z_1^* \\ &= a(z_2^*) - b(z_2^*)z_1^* = c(z_1^*) - d(z_1^*)z_2^* \end{aligned}$$

with $a_1, a_2, a_3, a_4 \in \mathbb{Z}^4 - \{0\}^4$, $a(z_2^*) = a_1 - a_2z_2^*$, $b(z_2^*) = b_1 - z_2^*b_2 \in \mathcal{O}_{\mathbb{K}_2}$ and $c(z_1^*) = a_1 - z_1^*b_1$ and $d(z_1^*) = a_2 - z_1^*b_2 \in \mathcal{O}_{\mathbb{K}_1}$. The relative norm $N_{\mathbb{F}/\mathbb{K}_2}[x] \in \mathcal{O}_{\mathbb{K}_2}$ is detailed in (11) with the trace expression being detailed in (12). Assume that $\exists x \in \mathbb{F}$ such that $N_{\mathbb{F}/\mathbb{K}_2}[x] = 2$, then,

$$\begin{aligned} N_{\mathbb{K}_1/\mathbb{Q}}[c(z_1^*)] + N_{\mathbb{K}_1/\mathbb{Q}}[d(z_1^*)] &= 2 \qquad (15) \\ \mathrm{Tr}_{\mathbb{K}_1/\mathbb{Q}}[c(z_1^*)\bar{d}(z_1^*)] - 5N_{\mathbb{K}_1/\mathbb{Q}}[d(z_1^*)] &= 0 \qquad (16) \end{aligned}$$

Define $k \in \mathbb{Z}$ as,

$$k \triangleq N_{\mathbb{K}_1/\mathbb{Q}}[d(z_1^*)],$$

the norm of the algebraic number $d(z_1^*) \in \mathcal{O}_{\mathbb{K}_1}$. Then, using (15) and (16),

$$\begin{aligned} N_{\mathbb{K}_1/\mathbb{Q}}[c(z_1^*)] &= 2-k, \\ \mathrm{Tr}_{\mathbb{K}_1/\mathbb{Q}}[c(z_1^*)\bar{d}(z_1^*)] &= 5k. \end{aligned}$$

Let $p(z_1^*)$ define the algebraic number,

$$p(z_1^*) = c(z_1^*)\bar{d}(z_1^*)$$

with $\bar{d}(z_1^*) = a_2 + \bar{z}_1^* b_2 \in \mathbb{K}_1$ then, we can easily observe that,

$$N_{\mathbb{K}_1/\mathbb{Q}}[p(z_1^*)] = N_{\mathbb{K}_1/\mathbb{Q}}[c(z_1^*)] N_{\mathbb{K}_1/\mathbb{Q}}[d(z_1^*)].$$

Finally, if $\exists\ x = c(z_1^*) - d(z_1^*) z_2^* \in \mathbb{F}$ such that $N_{\mathbb{F}/\mathbb{K}_2}[x] = 0$, this means that there exists an algebraic integer $p(z_1^*) = c(z_1^*)\bar{d}(z_1^*) \in \mathbb{K}_1$ such that,

$$\begin{aligned} \mathrm{Tr}_{\mathbb{K}_1/\mathbb{Q}}[p(z_1^*)] &= 5k, && (17) \\ N_{\mathbb{K}_1/\mathbb{Q}}[p(z_1^*)] &= k(2-k). && (18) \end{aligned}$$

The polynomial associated to this algebraic number $p(z_1^*) \in \mathbb{K}_1$ is,

$$\begin{aligned} \mu(X) &= (X-p)(X-\bar{p}) \\ &= X^2 - \mathrm{Tr}_{\mathbb{K}_1/\mathbb{Q}}[p(z_1^*)]X + N_{\mathbb{K}_1/\mathbb{Q}}[p(z_1^*)]. \end{aligned}$$

Consequently,

$$\mu(X) = X^2 - 5k\ X + k(2-k),$$

with $k \in \mathbb{Z}$ being a norm in $\mathbb{K}_1$. The discriminant of $\mu(X)$ is,

$$\Delta_\mu(k) = k(29k-8).$$

The algebraic number $p(z_1^*)$ is in $\mathcal{O}_{\mathbb{K}_1}$ if and only if the roots of $\mu(X)$ are in $\mathcal{O}_{\mathbb{K}_1}$.This is verified iff one of the following conditions is satisfied:

1) $k(2-k)$ is equal to zero $\Leftrightarrow$ ($k = 0$ or $k = 2$). Then,

$$p(z_1^*) = 5k \in \mathcal{O}_{\mathbb{K}_1}.$$

The solution $k = 2$ is impossible as k defines a norm in $\mathbb{K}_1$ and 2 is not a norm in $\mathbb{K}_1$. If $k = 0$, then, $N_{\mathbb{K}_1/\mathbb{Q}}[c(z_1^*)] = 2$ which is also impossible [25].

2) $\Delta_\mu(k)$ is a perfect square number n_1^2 with $n_1 \in \mathbb{Z}$ and $(n_1 + 5k)$ even. Then,

$$p(z_1^*) = \frac{1}{2}(5k + n_1) \in \mathcal{O}_{\mathbb{K}_1}.$$

If n_1 is a prime number then, $\Delta_\mu(k)$ is a perfect square if and only if $k = 21k-8$ which is impossible for $k \in \mathbb{Z}$. if n_1 is not a prime number $n_1 = \gamma_1\gamma_2$ with $\gamma_1, \gamma_2 \in \mathbb{Z}$, then, $\Delta_\mu(k)$ is a perfect square if and only if ($k = 29k-8$) or ($29k-8 = \gamma_1$ and $k = \gamma_1\gamma_2^2$) or ($29k-8 = \gamma_1\gamma_2^2$ and $k = \gamma_1$). This is equivalent to have ($5k = 2$) or ($k = \gamma_1\gamma_2^2$ and $(29\gamma_2^2 - 1)\gamma_1 = 8$) or ($k = \gamma_1$ and $(29 - \gamma_2^2)\gamma_1 = 8$). The only possibility to get a perfect square discriminant in $\mathbb{Z}$ is obtained for $\gamma_1 = 2$, $\gamma_2 = 5$ and $k = 2$. The discriminant $\Delta_\mu(2 \times 5^2) = 10^2$. We can also check that $(n_1 + 5k)$ is even.
The solution $k = 2$ is impossible as k defines a norm in $\mathbb{K}_1$ and 2 is not a norm in $\mathbb{K}_1$.

3) $\Delta_\mu(k)$ is in the form of $5n_2^2$ with $n_2 \in \mathbb{N}$ with $(n_2 + 5k)$ even. Then,

$$p(z_1) = \frac{n_2 + 5k}{2} - n_2 z_1^*$$

If n_2 is a prime number then, $\Delta_\mu(k) = 5n_2^2$ iff ($k = 5$ and $29k - 8 = n_2^2$) or ($k = 5n_2$ and $29k - 8 = n_2$) or ($29k - 8 = 5$ and $k = n_2^2$) or ($29k - 8 = 5n_2$ and $k = n_2$) which is impossible for $k \in \mathbb{Z}$ (2 is not a norm in $\mathbb{K}_2$ [25]).
if n_2 is not a prime number $n_2 = \gamma_1\gamma_2$ with $\gamma_1, \gamma_2 \in \mathbb{Z}$, then, $\Delta_\mu(k) = 5n_2^2$ iff in addition the prime case condition we have the choice of ($29k-8 = 5\gamma_1$ and $k = \gamma_1\gamma_2^2$) or ($29k-8 = 5\gamma_1\gamma_2^2$ and $k = \gamma_1$) or ($29k-8 = \gamma_1$ and $k = 5\gamma_1\gamma_2^2$) or ($29k - 8 = \gamma_1\gamma_2^2$ and $k = 5\gamma_1$). This is equivalent to have ($k = \gamma_1\gamma_2^2$ and $(29\gamma_2^2 - 5)\gamma_1 = 8$) or ($k = \gamma_1$ and $(29 - 5\gamma_2^2)\gamma_1 = 8$) or ($k = 5\gamma_1\gamma_2^2$ and $(29 \times 5 \times \gamma_2^2 - 1)\gamma_1 = 8$) or ($k = 5\gamma_1$ and $(29 \times 5 - \gamma_2^2)\gamma_1 = 8$). The only possibility to get a discriminant in the form is $\gamma_2 = \pm 12$, $\gamma_1 = 8$ and $k = 8 \times 5$ and $\Delta_\mu(8) = 96^2 \times 5$. We can also check that $(n_1 + 5k)$ is even.
The solution $k = 8 \times 5$ is impossible as k defines a norm in $\mathbb{K}_1$ and 8 is not a norm in $\mathbb{K}_1$, [25].

This means that is impossible to define $c(z_1^*)$ and $d(z_1^*)$ for which $x = c(z_1^*) - z_2^* d(z_1^*)$ has a norm $N_{\mathbb{F}/\mathbb{K}_2}[x] = 2$.

To prove the second statement, the same reasoning should be followed. We need to show that if $x \neq 0$, it is impossible to find $x \in \mathbb{F}$ such that $N_{\mathbb{F}/\mathbb{K}_2}[x] = 0$. Assume that there exists $x \in \mathbb{F}$ then $N_{\mathbb{F}/\mathbb{K}_2}[x] = 0$ if and only if:

$$\begin{aligned} N_{\mathbb{K}_1/\mathbb{Q}}[c(z_1^*)] + N_{\mathbb{K}_1/\mathbb{Q}}[d(z_1^*)] &= 0 && (19) \\ \mathrm{Tr}_{\mathbb{K}_1/\mathbb{Q}}[c(z_1^*)\bar{d}(z_1^*)] - 5N_{\mathbb{K}_1/\mathbb{Q}}[d(z_1^*)] &= 0 && (20) \end{aligned}$$

Following the same reasoning, the algebraic number $p(z_1^*) = c(z_1^*)\bar{d}(z_1^*)$ in $\mathcal{O}_{\mathbb{K}_1}$ has a characteristic polynomial,

$$\mu'(X) = X^2 - 5k^2 X - k^2,$$

with $k \triangleq N_{\mathbb{K}_1/\mathbb{Q}}[d(z_1^*)]$. The roots of $\mu'(X)$ satisfying constraints (19) and (20) are in the form $p(z_1^*) = z_2^* k$ or $\bar{z}_2^* k$ and are in $\mathcal{O}_{\mathbb{K}_1}$ if and only if $k = 0$ that leads to $x = 0$.

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