LaMI

Laboratoire de Méthodes Informatiques

Games networks and elementary modules

M. Manceny, F. Delaplace

e-mail : {mmanceny, delapla}@lami.univ-evry.fr

Rapport de Recherche n^o 120-2005

Novembre 2005

CNRS - Université d'Evry–Val d'Essonne
Cours Monseigneur Roméro
F–91000 Evry Cedex France

Games networks and elementary modules

Matthieu Manceny and Franck Delaplace

LaMI, CNRS-UMR 8042, Université d'Évry Val-d'Essonne, 91025 Evry, France
emails: {delapla, mmanceny}@lami.univ-evry.fr

Abstract. In this paper we propose an original modular extension of game theory named *games network*. The objective of games networks is to provide a theoretical framework which suits to modular dynamics resulting from different local interactions between various agents and which enables us to describe complex system in a modular way. Games networks describes situations where an agent can be involved in several different games, with several different other agents at the same time.
However, several games networks can represent the same dynamics. We focus on the determination of a *global equilibria*, resulting from the composition of local Nash equilibria, which allows us to compute a *games network normal form*. This normal form emphasizes the *elementary modules* which compose the games network.
Keywords: complex systems, modularity, game theory, networks.

1 Introduction

Analysis of complex systems is often based on the studies of relationships between components instead of elements themselves. It is the case in post-genomic studies ([17, 21]). This puts the emphasize on the way to analyse interactions. From modeling standpoints, networks provide a suitable framework to describe interactions (edges) of components (vertices). With networks, the description remains static and it is mainly focused on the structural analysis of the properties of the system ([13]).

In order to improve the framework by including dynamical aspects for the analysis of interactions, we propose to mix two formalisms: network formalism and game theory. Game theory has been pioneered by von Neumann and Morgenstern to define a theoretical framework to model complex interactions between agents or players ([15]). Game theory provides a modeling framework to characterize complex interplays in a large variety of fields such as Biology ([10, 16]), Economy ([7, 11]), Computer Science ([19, 1]). For instance in social and economical fields, it aims at analyzing situations where agents take decisions with the consciousness that outcomes of their own choices depends on the others. Decisions aim at maximizing payoffs and choices are assumed to be the result of a rational behavior. The rationality is however a metaphor suited for human interactions. In biology, adaptation and Darwinian selection are preferred to motivate the strategies of the agents.

Schematically, *games networks* can be viewed as a "network of games and players" where players are connected to the games they participate to. The representation corresponds to a bipartite graph where two categories of nodes are available: nodes representing players and nodes representing games. By using games networks, we describe the interactions as a set of modular activities where each games represents a module of interactions. Module description differs from usual representation such circuit devices, because components may belong to different modules. Components are reused for different modules.

In post-genomic, recent analysis on gene expression ([20]) appear to confirm this fact if we assume that each module corresponds to a coordinated set of responses to a stress. So, module finding is shifted from components to interactions.

Finding modules remains a challenging problem. The challenge relies on the relevance of the functional analysis deduced from the modular design proposed by the analysis. Modular analysis is important for biological applications such as drug design because its definition relies on an association between a support and a function. Hence, modules help identify targets for drugs. Whatever the description of a module might took, they share common general properties:

- *Generative*: each module is constitutive of a system of which it defines a building block. From the assembly of the modules, the system is formed and acquires its properties.
- *Functional*: subject to its unicity, the deterioration of a module leads to the loss of the properties assigned to the module.
- *Elementary*: this property refers to the atomicity of a module, that is, the impossibility to extract a sub module from a module.

Generative. Games networks theory provides a framework for (biological) dynamics based module analysis by describing the complexity of the interplays by games which are assimilated to modules. Essentially, modular dynamics relies on locality assumptions (represented by games). From the local properties of games, such as local equilibria, we compute global equilibria of a system by "assembling" each compatible equilibria (section 5.3).

Functional. In games networks, relationships between games and functions are determined by the modeling. Functions are described by interplays described in a game. Local (Nash) equilibria is then determined from each game. And the removal of a game induces the loss of the equilibria associated to a game.

Elementary. However the description may not represent a basic module, according to the previous properties, because the initial description may not be necessary elementary. Indeed, the property relies on the assumption that a game cannot be splinted into two sub-games. This can be hard to deal with during the design of the model. Hence, we propose an algorithm to automatically decompose into elementary modules. The automatic decomposition highlights new games structure of the former network and sometimes reveal another view of the system.

The paper is organized as follows: Section 2 deals with related work. Section 3 presents notations and general definitions used in this article. Section 4 briefly recalls the main result on strategic game theory. Section 5 presents the extension of strategic games to games networks and define global equilibrium at the scale of the whole network. Section 6 deals with structural modifications, that is the notion of equivalence between two games networks, and the operators that allow us to modify which agents participate to which games. Section 7 is interested in finding a games network normal form, that is a network composed of elementary modules (modules which can not be separated into smaller modules). We define an separation algorithm based on the notion of dependence to automatically computed normal form. We conclude in section 8.

2 Related work

Games with local interactions have been introduced to provide a framework to express locality which reduce the complexity of the Nash equilibria computation.

Indeed, research of the steady states of a game, and so computation of Nash equilibria, is certainly one of the most studied field in game theory. Moreover, McKelvey and McLennan ([12]) note that the computation of Nash equilibria in n-players games is much harder, in many important ways, that the computation in 2-players games. In games with local interactions, games are no longer considered in their globality, but through the local interactions between the players.

La Mura, in [8], to treat multi-agent decision problems, introduces a new game representation, more structured and more compact than classical representations in game theory. Considering the strategic separabilities in its representation, La Mura presents convergence methods to compute Nash equilibria.

Kearns, Littman and Singh in [5] introduce a compact graph-theoretic representation for multi-party game theory. Their main result is an efficient algorithm for computing approximate Nash equilibria in one-stage games represented by trees or sparse graphs.

Interested in Bayesian networks and in the locality of interactions, Koller and Milch in [6] propose a representation language for general multi-player games named Multi-Agent Influence Diagrams. They insist on the importance of dependence relationship among variables to detect structures in games and decrease the computational cost of finding Nash equilibria.

In this paper, we focus on interactions localized to a given process. Our games network representation, compared to La Mura, is not another game-theoretic representation but an extension of strategic representation. The closest representation is that of Kearns, Littman and Singh. However, in quite a some way as Koller and Milch, we are interested in the influence of the network organization, in terms of dependences between agents. We more particularly focus on the research of elementary modules which compose a game.

3 Notations and definitions

In the paper, we use the following notations.

- $[a : b] = \{i \in \mathbb{Z} | a \leq i \leq b\}$ denotes a discrete interval bounded by a and b.

Let A be a set, we note:

- $|A|$, the cardinal of A
- if $i \in A$, i also denotes the singleton $\{i\}$ if it is required by the context of the operation
- we consider the lifted version $A_{\mathbf{lift}} = A + \{\perp\}$ where the element *Bottom* denoted by $\perp$ is added to A
- Let $X \subseteq A^n_{\mathbf{lift}}, n \geq 1$, we denote by $\lceil X \rceil$ the set of profiles (or vectors) of X where each profile does not contain $\perp$, *i.e.*:

$$\lceil X \rceil = \{c \in X | \forall i \in [1 : n], c_i \neq \perp\}$$

Let $C = \{C_i\}_{i \in A}$ be a set of sets, we note:

- $C_{-j} = \times_{i \in A-j} C_i, \forall j \in A$
- $C_A = \times_{i \in A} C_i$
- $C^*_A = \bigcup_{X \subseteq A} C_X$, the set of all $k-$uples of C with $0 \leq k \leq n$

– Given a profile (or vector) $c \in C_A$
  - $c_{-i} = (c_1, \cdots, c_{i-1}, c_{i+1}, \cdots, c_n) \in C_{-i}$; this excludes the i^{th} component of a profile.
  - $(c_{-i}, \mathbf{c}_i) = (c_1, \cdots, c_{i-1}, \mathbf{c}_i, c_{i+1}, \cdots, c_n) \in C_A$; the notation distinguishes the i^{th} component of the profile from the others. This notation is extended to sets of indices, $(c_{-X}, c_X), X \subset A$.

Definition 1 ($\oplus$ operator). *We define the operator* $\oplus : A_{\mathbf{lift}} \times A_{\mathbf{lift}} \mapsto A_{\mathbf{lift}}$ *as follows:*
$\forall \alpha \in A_{\mathbf{lift}}, \alpha \oplus \bot = \bot \oplus \alpha = \alpha,$
$\forall \alpha \in A_{\mathbf{lift}}, \alpha \oplus \alpha = \alpha,$
$\forall (\alpha, \alpha') \in A^2_{\mathbf{lift}}, \alpha \neq \alpha' \Rightarrow \alpha \oplus \alpha' = \bot.$
The extension of the $\oplus$ operator to profiles and set of profiles is defined as follows:
$\forall (c, c') \in C_A^2, c \oplus c' = (c_i \oplus c'_i)_{i \in 1:n},$
$\forall (C, C') \in 2^{C_A}, C \oplus C' = \{c \oplus c' | c \in C, c' \in C'\}.$

Example 1 ($\oplus$ operator).

$$(a, b, c) \oplus (a, d, \bot) = (a, \bot, c)$$

$$\{(a, b, c), (a, d, c)\} \oplus \{(a, d, \bot), (x, y, c)\} = \{(a, \bot, c), (a, d, c), (\bot, \bot, c)\}$$

Definition 2 (nth operator). **nth** *denotes the rank of an integer in an integer subset according to the natural order*

$$\mathbf{nth} : \mathbb{N}^+ \times 2^{\mathbb{N}^+} \mapsto \mathbb{N}^+, \mathbf{nth}(i, A) = \begin{cases} |\{j \in A \mid j \leq i\}| & \text{if } i \in A \\ \text{undefined} & \text{otherwise} \end{cases}$$

*Example 2 (**nth** operator).* $\mathbf{nth}(5, \{1, 3, 5, 7\}) = 3$.

Definition 3 (Scatter). *We define the* Scatter operator *as follows:* $c \uparrow_A^B : C_A^* \times 2^{\mathbb{N}^+} \times 2^{\mathbb{N}^+} \mapsto (C_A^*)_\bot,$

$$\forall i \in [1 : \max(B)], (c \uparrow_A^B)_i = \begin{cases} c_{\mathbf{nth}(j,A)} & \text{if } \exists j \in A \cap B \text{ such that } \mathbf{nth}(j, B) = i \\ \bot & \text{otherwise} \end{cases}$$

Note: if $B = [1 : n]$ is an interval, and $A \subseteq B$, the definition becomes:

$$\forall i \in [1 : n], (c \uparrow_A^B)_i = \begin{cases} c_{\mathbf{nth}(i,A)} & \text{if } i \in A \\ \bot & \text{otherwise} \end{cases}$$

Example 3 (Scatter). For instance, given the following profile (a, b, c, d), we have:

$$(a, b, c, d) \uparrow_{\{1,3,7\}}^{[1:8]} = (a, \bot, b, \bot, \bot, \bot, c, \bot)$$

$$(a, b, c, d) \uparrow_{\{1,3,7\}}^{\{1,2,3,7,8\}} = (a, \bot, b, c, \bot, \bot, \bot, \bot)$$

$$(a, b, c, d) \uparrow_{\{1,3,5,17\}}^{\{1,2,8\}} = (a, \bot, \bot, \bot, \bot, \bot, \bot, \bot)$$

4 Strategic Games

In this section we give definitions of game theory used in the article. The reader may refer to the books [18, 14, 4] for a complete overview of game theory and its applications.

4.1 Definition of a strategic game

Strategic game is a model of interplays where each agent chooses its plan of action (or strategy) once and for all, and these choices are made simultaneously. Moreover, each agent is rational and perfectly informed of the payoff function of other agents. Thus, they aim at maximizing their payoffs while knowing the expectation of other agents.

Definition 4 (Normal or Strategic Representation). *A strategic game $\mathit{\Gamma}$ is a $3-$uple $\langle A, C, u\rangle$ where:*

- *A is a set of players or agents.*
- *$C = \{C_i\}_{i\in A}$ is a set of strategy sets. Each C_i represents the set of the m_i strategies available for agent i, $C_i = \{c_i^1, \cdots, c_i^{m_i}\}$.*
- *$u = (u_i)_{i\in A}$ is a vector of functions. Each $u_i : C \mapsto \mathbb{R}, i \in A$ represents the payoff function of the agent i.*

In order to conveniently combine sets of strategies, we define the strategy as follows:

Definition 5 (Set of Strategies). *Let $\langle A, C, u\rangle$ be a strategic game, let $\mathit{\Phi}^*$ be a set of labels, The set of strategies $C = \{C_i\}_{i\in A}$ are defined as follows $\forall i \in A, C_i = \{(i, \varphi) | \varphi \in \mathit{\Phi}^*\}$.*

By this definition, the fact that agents share the same strategies do not interfere in the union of sets of strategies.

4.2 Mixed (or randomized) strategies

Given a strategic game $\mathit{\Gamma} = \langle A, C, u\rangle$, a *mixed-strategy*[1] for any player i is a probability distribution over C_i. We let $\mathit{\Delta}(C_i)$ denote the set of all possible mixed strategies for player i.

$$\mathit{\Delta}(C_i) = \{(p_j)_{j\in[1:m_i]} | \forall j \in [1 : m_i], 0 \leq p_j \leq 1 \wedge \sum_{j=1}^{m_i} p_j = 1\}$$

A *mixed-strategy profile*[2] σ is any vector that specifies one mixed strategy $\sigma_i \in \mathit{\Delta}(C_i)$ for each agent $i \in A$. We let $\mathit{\Delta}(C)$ denotes the set of all possible mixed-strategy profiles.

$$\mathit{\Delta}(C) = \times_{i\in A} \mathit{\Delta}(C_i)$$

For any mixed-strategy profile $\sigma \in \mathit{\Delta}(C)$, let $u_i(\sigma)$ denotes the payoff for player i.

$$u_i(\sigma) = \sum_{c\in C} (\prod_{j\in A} \sigma_j(c_j)) u_i(c), \forall i \in A$$

[1] If the distribution is such that only one probability is different to 0, then the mixed-strategy is called *pure strategy.*

[2] If the strategy of each player is pure, the profile is said to be pure.

4.3 Nash equilibrium

Nash equilibrium is a central concept of game theory. This notion captures the steady states of the play of a strategic game in which each agent holds the rational expectation about the other players behavior. A *mixed Nash equilibrium* is defined as follows:

Definition 6 (Mixed Nash equilibrium of a strategic game). *Let $\langle A, C, u\rangle$ be a strategic game, and $\sigma^* \in \Delta(C)$ a mixed-strategy profile. σ^* is a* mixed Nash equilibrium[3] *iff:*

$$\forall i \in A, \forall \sigma_i \in \Delta(C_i), u_i(\sigma^*_{-i}, \sigma_i) \leq u_i(\sigma^*_{-i}, \sigma^*_i)$$

In other words, *no agent can unilaterally deviate of a mixed Nash equilibrium without decreasing its payoff.*

Definition 7 (Set of mixed Nash equilibria). *Let $G = \langle A, C, u\rangle$ be a game, we define* $\mathbf{mne}(G)$*, the set of mixed Nash equilibria for G:*

$$\mathbf{mne}(G) = \{\sigma^* \in \Delta(C) | u_i(\sigma^*_{-i}, \sigma_i) \leq u_i(\sigma^*_{-i}, \sigma^*_i), \forall i \in A, \forall \sigma_i \in \Delta(C_i)\}$$

5 Games Network

Games networks correspond to an extension of game theory which defines *modular interactions* localized to different subsets of agents. Each module corresponds to a specific game defined by a payoff function. Parameters of the payoff function are strategies of agents involved in the game. Agents are shared between different modules and played different games in parallel. However, they have the same set of strategies for every games they played. In a games network, several games are combined to form a more general structure of network. In this section, we address the main definitions of a games network. The reader may refer to [2] for a more complete overview.

5.1 Definition of a Games Network

The definition of a games network mainly consists of defining a set of agents connected to a set of games.

Definition 8 (Games Network). *A games network is a $3-$uple $\langle \mathcal{A}, C, \mathcal{U}\rangle$ where*

- *$\mathcal{A}$ is a set of agents or players.*
- *$C = \{C_i\}_{i\in\mathcal{A}}$ is a set of sets of strategies.*
- *$\mathcal{U} = \{\langle A, u\rangle\}$ is a set of* game nodes *where each $A \subseteq \mathcal{A}$ is a set of agents and $u : A \times C_A \mapsto \mathbb{R}$ is a set of payoff functions such that $u = \{u_i : C_A \mapsto \mathbb{R}\}_{i\in A}$.*

A games network offers a synthetic representation to define the different interplays between several players. The structure $\langle A, u\rangle$ totally determines a game played by a subset of agents since it useless to include the strategies which are the same for any agent of the network. A games network is represented by a bipartite graph $\langle \mathcal{A}, \mathcal{U}, E\rangle, E \subseteq \mathcal{A} \times \mathcal{U}$ where an edge $(i, \langle A, u\rangle)$ is a member of E if and only if $i \in A$ (See fig. 1 for an illustration of a "4-agents/3-games" games network).

[3] If the profile is pure, we speak about *pure Nash equilibrium.*

5.2 Restriction

A game node can be viewed as a sub game of a larger game played by the whole agents of the network. To focus on an arbitrary sub game, we equip the theory with the *restriction operator* which restricts a mixed-strategy profile to relevant values according to a subset of agents, named *support* of the sub game. A profile of values defined by a restriction is considered as a *local profile* of a subset of agents. Whatever the values associated to other agents are, they will not be considered for a local profile.

Definition 9 (Mixed-strategy Profile Restriction). *Let $\mathcal{A} = [1 : n]$ be a discrete interval representing a set of agents, let $C = \{C_i\}_{i \in \mathcal{A}}$ be a set of strategy sets. Given a mixed-strategy profile $\sigma \in \Delta(C)$*[4]*, we define its restriction to a subset $A \subseteq \mathcal{A}$, denoted by $\sigma{\downarrow_A}\colon \Delta(C) \times \mathbf{2}^{A} \mapsto \Delta(C)_{\mathbf{lift}}$, as follows*[5] *:*

$$(\sigma{\downarrow_A})_i = \begin{cases} \sigma_i & \text{if } i \in A \\ \bot & \text{otherwise} \end{cases}$$

We extend the restriction operator by removing bottom elements ($\bot$) of the profile, but the order of the other values is conserved in the resulting profile. We note the composition of the removals and restriction operation as follows: $\sigma \Downarrow_X$

The restriction is obviously extended to a set of mixed-strategy profiles by applying the operation to every elements.

Example 4. Let $\mathcal{A} = [1 : 4]$ and $\sigma = (\sigma_1, \sigma_2, \sigma_3, \sigma_4)$. Let $A = \{1, 3\}$, we have $\sigma \downarrow_A = (\sigma_1, \bot, \sigma_3, \bot)$ and $\sigma \Downarrow_A = (\sigma_1, \sigma_3)$.

The restriction applied to mixed-strategy profiles will be used in section 5.3 to put the focus on a sub part of a profile which corresponds to a game node.

5.3 Mixed Games network equilibrium

We define *global equilibria* at the scale of the games network. Such an equilibrium is named the *mixed games network equilibria* (**MGne**). A games network equilibrium corresponds to a compatible association of local equilibria. We assume that agents follow the *single played strategy* rule, that is an agent plays the same strategy for every connected games. The definition of **MGne** can of course be applied to the whole network, but the restriction to a subset of game nodes allow us to define regions where equilibria are compatible.

Definition 10 (Mixed Games Network Equilibrium). *Let $\Gamma = \langle \mathcal{A}, C, \mathcal{U} \rangle$ be a games network, let $\sigma^* = (\sigma_1, \cdots, \sigma_n) \in \Delta(C)$ be a mixed-strategy profile of every agents*[6]*. σ^* is a* mixed games network equilibrium *of a subset $U \subseteq \mathcal{U}$ (noted $\sigma^* \in \mathbf{MGne}_{\Gamma}(U)$) iff:*

$$\forall \langle A, u \rangle \in U, \sigma^* \Downarrow_A \text{ is a mixed Nash equilibrium of the game } \langle A, (C_i)_{i \in A}, u \rangle$$

Theorem 1 allows us to determine all the global equilibria of a games network.

[4] Recall that $\Delta(C)$ denotes the set of all possible mixed-strategy profiles
[5] $\bot$ stands for an irrelevant value
[6] Recall that by convention $|\mathcal{A}| = n$.

Theorem 1 (Largest Set of Global Equilibria). *Let $\Gamma = \langle \mathcal{A}, C, \mathcal{U} \rangle$ be a games network, let $U \subset \mathcal{U}, U = \{g_i = \langle A_i, u_i \rangle\}$ be a set of game nodes, and let $A = \bigcup_i A_i$. Then, the largest set of Mixed Games network equilibria for game nodes of U is*[7]*:*

$$\mathbf{MGne}_{\Gamma}(U) = \left\lceil \bigoplus_i \mathbf{mne}(g_i) \uparrow^{A}_{A_i} \right\rceil$$

5.4 An example of games network

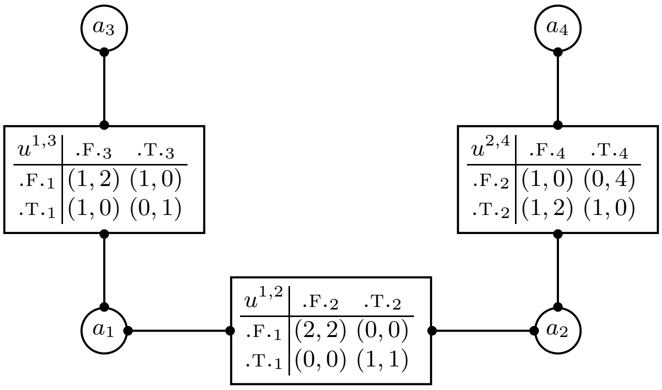


Fig. 1. Games network from section 5.4

Let us consider $\Gamma = \langle \mathcal{A}, C, \mathcal{U} \rangle$ the games network of fig. 1. We have:

- $\mathcal{A} = \{a_1, a_2, a_3, a_4\}$, the agents
- $C_i = \{.\text{F}._i, .\text{T}._i\}, \forall i \in \mathcal{A}$, the strategies of the agents
- $\mathcal{U} = \{\langle A_{1,3}, u^{1,3} \rangle, \langle A_{1,2}, u^{1,2} \rangle, \langle A_{2,4}, u^{2,4} \rangle\}$, the game nodes where $A_{1,3} = \{a_1, a_3\}$, $A_{1,2} = \{a_1, a_2\}$, $A_{2,4} = \{a_2, a_4\}$ and the payoffs functions are shown in fig. 1.

To compute the **MGne** of Γ, let us compute the **mne** of each sub game.
$\mathbf{mne}_{1,3} = \mathbf{mne}(\langle A_{1,3}, u^{1,3} \rangle) = \{((1,0),(1,0))\ ;\ ((\frac{1}{3},\frac{2}{3}),(1,0))\}$
$\mathbf{mne}_{1,2} = \mathbf{mne}(\langle A_{1,2}, u^{1,2} \rangle) = \{((1,0),(1,0))\ ;\ ((0,1),(0,1))\ ;\ ((\frac{1}{3},\frac{2}{3}),(\frac{1}{3},\frac{2}{3}))\}$
$\mathbf{mne}_{2,4} = \mathbf{mne}(\langle A_{2,4}, u^{2,4} \rangle) = \{((0,1),(1,0))\ ;\ ((\frac{1}{3},\frac{2}{3}),(1,0))\}$

Thus, we can compute the Mixed Games network equilibria of Γ:

$$\mathbf{MGne}_{\Gamma}(\mathcal{U}) = \{((\frac{1}{3},\frac{2}{3}),(\frac{1}{3},\frac{2}{3}),(1,0),(1,0))\}$$

6 Structural modifications

The definition of games networks allows the combination of several games into a single network. This puts the emphasis on the way that the network structure is determined, because different structures can be proposed to model the same situation.

[7] Definitions of $\lceil\ \rceil$, $\bigoplus$ and $\uparrow^{A}_{A_i}$ are given in section 3, page 3.

The definition of **MGne** seen in previous section provides the ability to define equivalence between different games networks in section 6.1. The equivalence opens on the possibility of establishing structural modifications of a games network. Operators allowing such modifications are detailled in section 6.2. We will see in section 6.3 that this possibility reveals the importance of an observer function used in the different operators.

6.1 Equivalence between Games Networks

Equivalence between two games networks is based on the equality of their equilibria. More precisely, the equivalence is based on the largest set of global sets of equilibria:

Definition 11 (MGne Equivalence). *Let* $\Gamma_1 = \langle \mathcal{A}_1, C_1, \mathcal{U}_1 \rangle$ *and* $\Gamma_2 = \langle \mathcal{A}_2, C_2, \mathcal{U}_2 \rangle$ *be two games networks such that* $\mathcal{A}_1 = \mathcal{A}_2, C_1 = C_2$*.* Γ_1 *and* Γ_2 *are equivalent, denoted by* $\Gamma_1 \equiv_{\mathbf{MGne}} \Gamma_2$*, if and only if* $\mathbf{MGne}_{\Gamma 1}(\mathcal{U}_1) = \mathbf{MGne}_{\Gamma 2}(\mathcal{U}_2)$

Informally, it means that both games networks have the same dynamics if we admit that equilibria represent steady states of the network.

6.2 Operators

Operators detailled here allow us to modify the structure of a games network. Restructuring games networks is expressed in terms of substituting game nodes by others. The join operation or, conversely, the separation are the basic operations for games networks reorganization.

However, the reorganization can be performed if the initial games network and that resulting of the reorganization are equivalent in the sense of the definition 11.

Substitution The operation of substitution is formally defines as follows:

Definition 12 (Substitution). *Let* $\Gamma = \langle \mathcal{A}, C, \mathcal{U} \rangle$ *be a games network, let* $U = \{g_i = \langle A_i, u^i \rangle\}, U \subseteq \mathcal{U}$ *be a set of game nodes, let* $U' = \{\langle A_{i'}, u^{i'} \rangle\}$ *be another set of game nodes such that* $\forall i', A_{i'} \subseteq \mathcal{A}$*, we define the substitution, denoted by* $\Gamma_{[U/U']}$ *as follows:.*

$$\Gamma_{[U/U']} = \langle \mathcal{A}, C, \mathcal{U} - U \cup U' \rangle$$

Join operation The *join operation* consists in joining two game nodes in one. It is formally defined as follows:

Definition 13 (Join according to ω). *Let* $\Gamma = \langle \mathcal{A}, C, \mathcal{U} \rangle$ *be a games network, let* $G_1 = \langle A_1, u^1 \rangle$ *and* $G_2 = \langle A_2, u^2 \rangle$ *be two game nodes of* Γ $(G_1 \in \mathcal{U}, G_2 \in \mathcal{U})$*, let* $\omega : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ *be a function, we define:* $G_1 \bigvee^{\omega} G_2 = \langle A_1 \cup A_2, u \rangle$ *with :*

$$\begin{aligned} \forall c \in C_{(A_1 \cup A_2)}, \; & \forall i \in A_1 - A_2 \; u_i(c) = {u^1}_i(c \Downarrow_{A_1}) \\ & \forall i \in A_2 - A_1 \; u_i(c) = {u^2}_i(c \Downarrow_{A_2}) \\ & \forall i \in A_1 \cap A_2 \; u_i(c) = \omega({u^1}_i(c \Downarrow_{A_1}), {u^2}_i(c \Downarrow_{A_2})) \end{aligned}$$

The join operation depends on a function ω. For instance, the maximum function $\max(v_1, v_2)$ can be a candidate for giving a concrete definition of $\bigvee$ operation. If no specific property on ω is required we omit it in the specification of the operation.

Separation Separation is the reciprocal operation of the join operation. It consists in spliting a game node in two others. However, we imposes that equilibria are preserved during the separation. The separation, according to a function ω, is defined as follows:

Definition 14 (Separation according to ω). *Let $\Gamma = \langle \mathcal{A}, C, \mathcal{U} \rangle$ be a games network, a game node $G = \langle A, u \rangle \in \mathcal{U}$ is said to be separable (according to ω) if:*

$$\begin{array}{l}\exists G_1 = \langle A_1, u^1 \rangle, \exists G_2 = \langle A_2, u^2 \rangle \text{ such that} \\ \quad G_1 \bigvee^{\omega} G_2 = G \text{ and } \mathbf{mne}(G_1 \bigvee^{\omega} G_2) = \mathbf{MGne}(\{G_1, G_2\})\end{array}$$

6.3 Structural modifications and importance of the observer

Join operation and separation provides a general condition to restructure games networks based on the preservation of the equilibria. A special attention is paid on the reciprocal operation of the join because it enables us to split a games network into another one composed of more elementary games. This leads to the following theorem which defines a basic condition to perform modifications of the network.

Theorem 2 (Restructuration with separation). *Let $\Gamma = \langle \mathcal{A}, C, \mathcal{U} \rangle$ be a games network, let $G \in \mathcal{U}$ be a game node, and let $G_1 = \langle A_1, u^1 \rangle, G_2 = \langle A_2, u^2 \rangle$ be two game nodes such that $A = A_1 \cup A_2$.*

If G is separable according to a function ω to (G_1, G_2) then we have:

$$\Gamma \equiv_{\mathbf{MGne}} \Gamma_{[G/\{G_1, G_2\}]}$$

Implicitly, the structural modifications are dependant on a particular function ω, called *observer* and used in join or separation. Different observers will not allow or provide the same structural modifications.

Whereas separation preserves equilibria, it is not the case with the join operation. Thus, considering a games networks, the resulting joint game can be very different according to the observer we use.

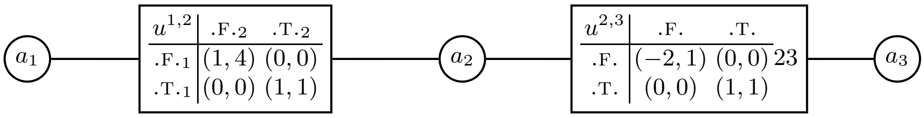
For example, let us consider the games network Γ from fig. 2 which is composed of two game nodes $g^{1,2}$ and $g^{2,3}$. We use the join operator on these two game nodes; we obtain Γ^M with the *Max* function as observer, and Γ^m with the *min* function. As we see on fig. 2, none of the games network Γ and its two joint games Γ^M and Γ^m has the same dynamics, *i.e.* the same equilibria.

On the other hand, separation preserves equilibria. So, if a games networks can be separate using two different observers, the resulting games networks have the same equilibria. However, observer influences the possibility of separation. Different observers allow different games to be separated.

7 Elementary modules

In this section, given a games network Γ, we are interested in a games network Γ', called *normal form*, which have the same equilibria as Γ, and which is composed of the smallest possible game nodes (in sense of number of agents involved in the game node). Games nodes of such a games network are called *elementary games* or *elementary modules*. The normal form presented in section 7.1 is dependent on a given function (as join operation

$$\Gamma = \langle \{a_1, a_2, a_3\}, \{.\mathrm{F}._i, .\mathrm{T}._i\}_{i\in[1:3]}, \{g^{1,2}, g^{1,3}\} \rangle$$



$$\mathbf{MGne}_{\Gamma} = \{ ((1,0),(1,0),(1,0)) \ ; \ ((0,1),(0,1),(0,1)) \}$$

$\Gamma^M = g^{1,2} \bigvee^{Max} g^{2,3}$

a_1	a_2	a_3	u_1^M	u_2^M	u_3^M
.F.1	.F.2	.F.3	1	4	1
.F.1	.F.2	.T.3	1	4	0
.F.1	.T.2	.F.3	0	0	0
.F.1	.T.2	.T.3	0	1	1
.T.1	.F.2	.F.3	0	0	1
.T.1	.F.2	.T.3	0	0	0
.T.1	.T.2	.F.3	1	1	0
.T.1	.T.2	.T.3	1	1	1

$\Gamma^m = g^{1,2} \bigvee^{\min} g^{2,3}$

a_1	a_2	a_3	u_1^m	u_2^m	u_3^m
.F.1	.F.2	.F.3	1	−2	1
.F.1	.F.2	.T.3	1	0	0
.F.1	.T.2	.F.3	0	0	0
.F.1	.T.2	.T.3	0	0	1
.T.1	.F.2	.F.3	0	−2	1
.T.1	.F.2	.T.3	0	0	0
.T.1	.T.2	.F.3	1	0	0
.T.1	.T.2	.T.3	1	1	1

$$\mathbf{MGne}_{\Gamma^M} \supset \mathbf{MGne}_{\Gamma} \cup \{ ((\tfrac{1}{4}, \tfrac{3}{4}), (\tfrac{1}{2}, \tfrac{1}{2}), (0,1)) \}$$

$$\mathbf{MGne}_{\Gamma^m} = \{ ((0,1),(0,1),(0,1)) \}$$

Fig. 2. Importance of the observer function

or separation operator in previous section). Section 7.2 extends the normal form to deal with a class of functions. Section 7.3 presents a new notion, *dependence*, which allows us to define, in section 7.4, an algorithm to compute a normal form.

7.1 Normal Form

Games network normal form is defined as follows:

Definition 15 (Normal Form according to a function). *Let Γ be a games network, $\omega : \mathbb{R}^2 \mapsto \mathbb{R}$ a function.*

*Γ is said to be $\omega-$*normal *if it is not separable according to ω.*

In normal form, each game node is called *elementary game* or *elementary module*.

A normal form can be computed by successive separations, that is each sub-game of a game is obtained by separation according to the considered function ω. When separation is applied, the agents are distributed in the two games resulting from separation. In this case, they result from the impact that separation has on the agents. According to definition 13, the problem is reduced to the way in which the payoff function of each game node is computed from the payoff function of the original game.

7.2 $\Omega-$Normal Form

Structural modifications may generate infinite alternatives of games networks from a given games network. For example, if we assume that ω selects the first argument regardless the value of the second one then given a game node $G = \langle A, u\rangle$, we have $G = G \bigvee^{\omega} G', G' = \langle A', u'\rangle$ providing $\mathbf{mne}(G) = \mathbf{mne}(G')$ and $A' \subseteq A$. Thus, without additional constraints there is no *a priori* unicity of the normal form.

Moreover, it seems also desirable that a normal form addresses a class of functions instead of a specific function because we obtain a more general process for the reorganization. Indeed, if we admit that ω formalizes the viewpoint of the observer, then, by addressing a class of the functions Ω, the reorganization is compatible with the viewpoints of all the observers of this class.

Among possible classes of functions, some of them appear to be more relevant for modeling. We address the computation of the normal form for *functions with neutral element* which are defined as follows:

Definition 16 (Function with Neutral Element). *Let Ω be the set of idempotent function with neutral element defined as follows:*

$$\Omega = \{\omega : \mathbb{R}^2 \mapsto \mathbb{R} | \exists e_\omega \in \mathbb{R}, \forall x \in \mathbb{R}, \omega(x, e_\omega) = \omega(e_\omega, x) = x\}$$

In the sequel, the neutral element will be denoted by e if we do not consider a specific function of Ω but a generic instance of them.

The extension of the normal form to Ω will be defined according to the properties commonly shared by every functions of the class, that is, the neutral property. It is based on a new definition of the join operator as follows:

Definition 17 (Ω-Join). *Let Ω be the class of functions defined in 16. Let $\Gamma = \langle \mathcal{A}, C, \mathcal{U}\rangle$ be a games network, let $G_1 = \langle A_1, u^1\rangle$ and $G_2 = \langle A_2, u^2\rangle$ be two game nodes of Γ ($G_1 \in \mathcal{U}, G_2 \in \mathcal{U}$), we define:*

$$G_1 \bigvee^{\Omega} G_2 = \langle A_1 \cup A_2, u\rangle$$

with :

$$\begin{aligned}
\forall c \in C_{(A_1 \cup A_2)}, \; & \forall i \in A_1 - A_2 \; u_i(c) = {u^1}_i(c \Downarrow_{A_1}) \\
& \forall i \in A_2 - A_1 \; u_i(c) = {u^2}_i(c \Downarrow_{A_2}) \\
& \forall i \in A_1 \cap A_2 \; u_i(c) = \begin{cases} {u^2}_i(c \Downarrow_{A_2})) \text{ if } {u^1}_i(c \Downarrow_{A_1}) = e_\omega \\ {u^1}_i(c \Downarrow_{A_1})) \text{ if } {u^2}_i(c \Downarrow_{A_2}) = e_\omega \\ \text{undefined} \quad \text{otherwise} \end{cases}
\end{aligned}$$

The definition of the join operator, is now compatible with any functions of the class. Hence the separation is the same whatever the function ω is. This provides the ability to compute a function regardless to the specificity of a specific function.

Definition 18 (Games Network $\Omega-$Normal Form).

Let Γ be a games network, Ω the set of functions with neutral element.
*Γ is said to be $\Omega-$*normal *if any game node is inseparable according to the Ω join operator.*

7.3 Dependence

With normal form, we are interested in finding elementary modules which composed the network. Intuitively, agents involved in the same elementary module are agents of the original network which are higly interacting. To precisely describe the interplays occurring in a game, we define the notion of *dependence* between agents. Informally, an agent is dependent on another if its payoffs are altered by the strategies of the other player.

Definition 19 (Agent dependence). *Let $\langle A, C, u\rangle$ be a strategic game, let $j, i \in A^2, i \neq j$ be two agents. j is said to be* dependent *on i, denoted by $i\delta_u j$, if:*

$$\exists c_i \in C_i, \exists c'_i \in C_i, \exists c_{-i} \in C_{-i}, u_j(c_{-i}, c_i) \neq u_j(c_{-i}, c'_i)$$

The dependences provide an overview of the interplays of the agents in a game without having carefully studying the payoff function. To get an abstraction of the dependences according to a game, we introduce a new representation named *the agent dependence graph*.

Definition 20 (Agent Dependence Graph). *Let $G = \langle A, C, u\rangle$ be a strategic game, the agent dependence graph $D_G = \langle A, E\rangle$ is a graph such that: $E = \{(i, j)|i\delta_u j\}$*

Definition 21 (Set of predecessors).
Let $G = \langle A, C, u\rangle$ be a strategic game. We denote by $\delta_u^-(j)$, $j \in A$, the set of predecessors of j in the dependence graph of game G, that is

$$\forall j \in A, \delta_u^-(j) = \{i \in A|i\delta_u j\}$$

The dependence relation for a game is extended to the dependence relation by considering a games network as follows:

Definition 22 (Dependence relation according to a games network). *Let $\Gamma = \langle \mathcal{A}, C, \mathcal{U}\rangle$ be a games network, let $i \in \mathcal{A}$ and $j \in \mathcal{A}$ be two agents,*

$$i\delta_{\mathcal{U}} j \Leftrightarrow \exists G = \langle A, u\rangle \in \mathcal{U} \text{ such that } i\delta_u j$$

(Definition of dependence graph is extended in the same way.)

Example 5. Let Γ be a games network, we consider the following game node

$$g = \langle\{a_1, a_2, a_3, a_4\}, u\rangle$$

where u is defined as follows:

a_1	a_2	a_3	a_4	u_1	u_2	u_3	u_4
.F.$_1$	.F.$_2$	.F.$_3$	.F.$_4$	0	0	1	1
.F.$_1$	.F.$_2$	.F.$_3$	.T.$_4$	0	0	1	0
.F.$_1$	.F.$_2$	.T.$_3$	.F.$_4$	0	0	0	1
.F.$_1$	.F.$_2$	.T.$_3$	.T.$_4$	0	0	0	0
.F.$_1$	.T.$_2$	.F.$_3$	.F.$_4$	1	2	1	0
.F.$_1$	.T.$_2$	.F.$_3$	.T.$_4$	1	2	1	1
.F.$_1$	.T.$_2$	.T.$_3$	.F.$_4$	1	2	0	0
.F.$_1$	.T.$_2$	.T.$_3$	.T.$_4$	1	2	0	1
.T.$_1$	.F.$_2$	.F.$_3$	.F.$_4$	2	1	0	1
.T.$_1$	.F.$_2$	.F.$_3$	.T.$_4$	2	1	0	0
.T.$_1$	.F.$_2$	.T.$_3$	.F.$_4$	2	1	1	1
.T.$_1$	.F.$_2$	.T.$_3$	.T.$_4$	2	1	1	0
.T.$_1$	.T.$_2$	.F.$_3$	.F.$_4$	0	0	0	0
.T.$_1$	.T.$_2$	.F.$_3$	.T.$_4$	0	0	0	1
.T.$_1$	.T.$_2$	.T.$_3$	.F.$_4$	0	0	1	0
.T.$_1$	.T.$_2$	.T.$_3$	.T.$_4$	0	0	1	1

From the table describing u, we can deduce the following dependencies: $a_1\delta_u a_3$, $a_2\delta_u a_4$, $a_1\delta_u a_2$ and $a_2\delta_u a_1$

The corresponding dependence graph is shown on fig. 3

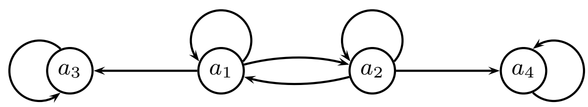


Fig. 3. Dependence graph for the games network from example 5

7.4 Algorithm

Many normal forms are possible given a games network. With the notion of dependence, we have find an algorithm which computes a specific normal form. The algorithm considers each game node as a network reduced to this node and computes a normal form with it. Then, the obtained networks will be assembled to obtain a normal form of the complete network.

Figure 4 presents the *separate* function which computes a normal form from a game node, and for a given function $\omega \in \Omega$.

Let $G = \langle A, u\rangle$ the starting game node and ω the observer function. First, the separate function research how many game nodes have to be created. The dependence graph is used to emphasize the interactions between agents and thus determine which agents participate to a same game node. The game nodes are defined by the agents which are involved in. For each agent, a game node which contains all its predecessors exists and, given two game nodes $g_1 = \langle A_1, u_1\rangle$ and $g_2 = \langle A_2, u_2\rangle$, we cannot have $A_1 \subseteq A_2$ or $A_2 \subseteq A_1$.

Once we have the game nodes, we have to compute the payoffs. Let $a \in A$ be an agent and g be a game node.

- If all the predecessors of a are in g, we can easily compute the payoffs for a, because none of the absent agents in g have any influence on a's payoffs. In fact, for any game $\langle A^*, C^*, u^*\rangle$, we have:

$$\forall \sigma, \sigma' \in \Delta(C^*)^2, \forall j \in A^*, \sigma \Downarrow_{\delta^-_{u^*(j)}} = \sigma' \Downarrow_{\delta^-_{u^*(j)}} \Rightarrow u^*_j(\sigma) = u^*_j(\sigma')$$

  Thus, given a pure profile c_g of g, each pure profile c_G of G such that the restriction of c_G to the agents of g equals c_g gives the same payoffs for a. The *pick* function in fig. 4 chooses one of these c_G profile.
- If at least one of the predecessors of a is not in G, we cannot compute a's payoff. Thus, we give e, the neutral element of ω, to a as payoff.

Example 6. According to the algorithm and from the dependence graph, we can deduce that the game node from example 5 is separated into three game nodes, each one having 2 agents. Figure 5 describes the resulting games network. Each game node is denoted by $g_{i,j} = \langle \{i, j\}, u\rangle$.

Being given a game node $\langle A, u \rangle$, we define:

δ^- : $A \mapsto \mathbf{2}^A$ the set of predecessors in the agent dependence graph
agent : $\mathbb{N} \mapsto \mathbf{2}^A$ the set of agents connected to the game node.
pick : $C_A \times (C_A \mapsto \mathbb{R}) \mapsto \mathbb{R}$,
pick(c', u) gives a value $u(c)$ such that the configuration c' is contained in c.

function **separate**($\langle A, u \rangle$: game node)
  $\mathcal{U}' := \emptyset$; $g := 0$;
  /**Computation of the number of game nodes to be created**/
  forall $i \in A$
    $g := g + 1$;
    agent$(g) := i \cup \delta_u^-(i)$;
  endforall
  $U = [1 : g]$;
  forall $g' \in [1 : g]$
    $U := U - \{g'' \in U | \mathbf{agent}(g'') \subset \mathbf{agent}(g') \vee (\mathbf{agent}(g') = \mathbf{agent}(g'') \wedge g'' < g')\}$;
  endforall

  /**Attribution of the payoffs**/
  forall $g \in U$
    forall $j \in \mathbf{agent}(g)$
      if $\delta_u^-(j) \cap \mathbf{agent}(g) = \delta_u^-(j)$ then
        forall $c \in C_{\mathbf{agent}(g)}$ $u_j^g(c) := \mathbf{pick}(c, u)$
      else
        forall $c \in C_{\mathbf{agent}(g)}$ $u_j^g(c) := e$
      endif
    endforall
      $\mathcal{U}' = \mathcal{U}' \cup \{\langle \mathbf{agent}(g), u^g \rangle\}$;
  endforall
  return $\mathcal{U}'$;

Fig. 4. Normal Form Algorithm for a Game Node

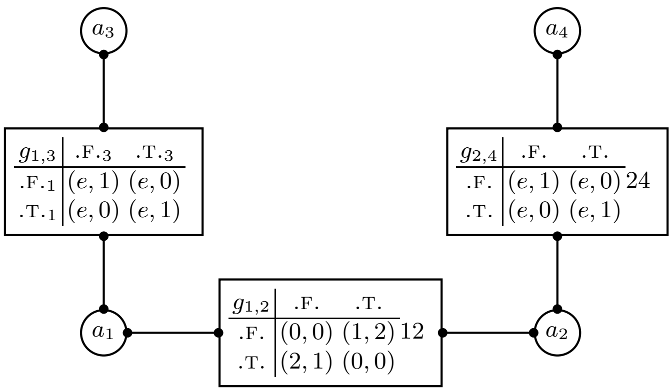


Fig. 5. Normal form of the games network from example 6

8 Conclusion

In this paper we have propose an extension of game theory, named *games networks*, which provides a framework to model complex systems in terms of sets of *interacting agents*. Whereas in game theory all the agents are interacting together, games networks allow us to define *local interactions* which help us understand the structure of complex systems.

In games networks, an agent can play several games with different sets of agents; each game represents local interactions. These interactions define a games network dynamics, which is caracterized by its observable states, *i.e.* its steady states. We have defined the notion of *global equilibria* which are steady states at the scale of the whole network, and which are a composition of local equilibria (Nash equilibria of the different games composing the network).

Different compositions of local interactions can provide the same global interactions. For that reason we have define a global-equilibria-based equivalence in order to compare two games networks. We have define structure modification operators (such as joint or separation) to transform a games network to another equivalent network. We have particularly focus on the definition of a *games network normal form* that is a network where each game can not be separated. The games in a normal form games network are called *elementary modules*. We define an algorithm which compute a normal form, using the separation operator. This algorithm is based on the notion of *dependence*, which allow us to precisely study the interactions of a network.

Games networks have been used to model biological complex systems. In [3], we deals with the Plasminogen Activation system (PAs). PAs is a process of signal transduction implied in the migration of cancer cells. With games networks, we have been able to model the system and to compute equilibria, which correspond to biological observable states: a promigratory state, and a non-migratory one.

As perspectives of this work, we plan to deal with other biological systems (as the λ phage for example). But some theoretical questions have to be answered such as the unicity of games network normal form or the use of class of functions (different of the functions with neutral elements). We are also interested in the dependence notion, more particularly in the link between self dependence agents (agents which depend on themselves) and the existence of global equilibria [9].

Acknowledgements

The authors would like to thank J.-L. Giavitto and O. Michel of the LaMI for their many discussions on complex systems. The authors thank C. Soulé of IHES, the G^3 research group members and especially F. Képès, for their stimulating discussions. At least, the authors thank G. Barlovatz of the DYNAMICS Team of LaMI (UMR 8042 CNRS) for the many discussions about the biological aspects of complex systems.

References

1. R. Alur, T.A. Henzinger, and O. Kupferman. Alternating-time temporal logic. In *Proceedings of the 38th IEEE Symposium on Foundations of Computer Science*, Florida, October 1997.

2. F. Delaplace and M. Manceny. Games network. Technical Report 101-2004, Laboratoire de Méthodes Informatiques (LaMI), CNRS-UMR 8042, Université d'Évry-Val d'Essonne, 2004.
3. F. Delaplace, M. Manceny, A. Petrova, M. Malo, F. Maquerlot, R. Fodil, D. Lawrence, and G. Barlovatz-Meimon. The "PAI-1 game": towards modelling the Plasminogen Activation system (PAs) dependent migration of cancer cells with the games network theory. In *Integrative Post Genomics (IPG)*, 2004.
4. R. Gibbons. *Game Theory for Applied Economists*. Princeton University Press, 1992.
5. Michael Kearns, Michael L. Littman, and Satinder Singh. Graphical models for game theory. In *Proceedings of the 17th Conference on Uncertaintly in Artificial Intelligence (UAI)*, pages 253–260, 2001.
6. Daphne Koller and Brian Milch. Multi-agent influence diagrams for representing and solving games. In *Proceedings of the 17th International Joint Conference on Artificial Intelligence (IJCAI)*, pages 1027–1034, 2001.
7. D.M. Kreps. *A Course in Microeconomic Theory*. Princeton University Press, 1990.
8. Pierfrancesco La Mura. Game networks. In *Proceedings of the 16th Conference on Uncertaintly in Artificial Intelligence (UAI)*, pages 335–342, 2000.
9. M. Manceny and F. Delaplace. On the relationships between interactions and equilibria in games networks. In *First Spain Italy Netherlands Meeting on Game Theory (SING)*, 2005.
10. J. Maynard Smith. *Evolution and the Theory of Games*. Cambridge Univ. Press, 1982.
11. R. D. McKelvey and A. McLennan. Computation of equilibria in finite games. In *Handbook of Computational Economics*, volume 1, pages 87–142. Elsevier, 1996. http://econweb.tamu.edu/gambit/.
12. Richard D. McKelvey and Andrew McLennan. Computation of equilibria in finite games. In *Handbook of Computational Economics*, volume 1, pages 87–142. Elsevier Science Pub Co, 1996.
13. R. Milo, S. Shen-Orr, S. Itzkovitz, N. Kashtan, D. Chklovskii, and U. Alon. Network motifs: Simple building blocks of complex networks. *Science*, 298:824–827, 2002.
14. R. B. Myerson. *Game Theory : Analysis of Conflict*. Harvard University Press, 1991.
15. J. Von Neumann and O. Morgenstern. *Theory of games and economic behavior*. Princeton University Press, Princeton, New Jersey, first edition, 1944.
16. M. A. Nowak and K. Sigmund. Evolutionary dynamics of biological games. *Sciences*, 303(6):793–799, februar 2004.
17. Zoltán N. Oltvai and Albert-László Barabási. Life's complexity pyramid. *Science*, 298:763–764, 2002.
18. M. J. Osborne and A. Rubinstein. *A Course in Game Theory*, volume 380. MIT Press, 1994.
19. C. H. Papadimitriou. Game theory and mathematical economics: a theoretical computer scientist's introduction. In *42nd IEEE Symposium on Foundations of Computer Science: Proceedings*, pages 4–8, 2001.
20. Eran Segal, Michael Shapira, Aviv Regev, Dana Pe'er, David Botstein, Daphne Koller, and Nir Friedman. Module networks: identifying regulatory modules and their condition-specific regulators from gene expression data. *Nature Genetics*, 34:166–176, June 2003.
21. D.M. Wolf and A. Arkin. Motifs modules and games in bacteria. *Current Opinion in Microbiology*, 6:125–134, 2003.