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Games Network

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Abstract. In this paper we present an extension of the game theory named *games networks*. The purpose of this extension is to define a framework for the specification of modular interactions, i.e. localised to groups of agents. Briefly, the games networks describe the situation where each player plays different games at the same time with several different players. It is graphically represented by a network (or graph) of games. We present this theoretical extension on strategic games. We more particularly focus on the determination of a global equilibrium from local Nash equilibria and on the structural reorganizations of a game while preserving the global equilibria. We propose an automatic reorganization of a games network which computes a normal form for a games network.

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1 Introduction

The game theory originated more than 50 year ago [13] to tackle problems involving interdependencies among several agents. It provides a modeling framework to characterize complex interplays in a large variety of fields such as Biology [8, 14], Economy [9, 10], Computer Science [16, 1]. For instance in social and economical fields, it aims at analyzing situations where agents take decisions with the consciousness that outcomes of their own choices depend on the others. The decisions aim at maximizing payoffs and the choices are assumed to be the result of a rational behavior. The rationality is however a metaphor suited for human interaction. In biology, adaptation and Darwinian selection are preferred to motivate the strategies of the agents.

A large number of results on game theory are centered on the *Nash equilibrium* [12]. A Nash equilibrium corresponds to an optimal *strategy* chosen by an agent to maximize its outcome while considering the rational behaviors of the other agents which also attempt to maximize their outcomes. No agent can unilaterally deviate of a Nash equilibrium without decreasing its outcome.

The game theory however corresponds to a local theory in the sense that it focuses on specific interactions of a given group of agents for the computation of Nash equilibria. To analyze the interactions on the scale of the entire system, it is often required to get a structured view of the system organization. The analysis of the properties of a system is often based on the study of the properties of sub-structures of the system. Each subpart is named a *module*. A module puts in correspondence a structure and characteristic properties of dynamics. Analyzing modular activities appears to be a central step on the understanding of a system because it determines an organization of a system from its sub-systems. For each sub-system, one may associate a particular function which takes part in the explanation of dynamics of the complete system. The resulting function is then viewed as a complex composition of functions handled by sub-systems. For instance in biology, this approach is developed to understand the interplays occurring in molecular networks [7, 17, 3, 20, 18]. Not only this approach makes it possible to characterize the organization of a system but it also makes it possible to reduce the cost of the Nash equilibrium computation by introducing two levels in the computation: local computations with groups of few agents, and their assembly giving the total equilibrium.

To tackle with the description of the modularity, we propose an original theoretical framework named *Games network*[1]. The games network theory extends the game theory by considering that agents can be involved in different games at the same time. Consequently, each game to which an agent participates must be considered to determine its strategy. Informally a games networks is determined by a set of agents playing to a *game node*. The network represents several games connected to agents. A games network can be conveniently drawn by a bipartite graph with two classes of nodes. A node of the first class represents games whereas a node of the the second represents agents

If we admit that the interactions between agents in a module are embodied by game node, an agent does not specifically belong to a single module but to different modules. For instance, in biology the definition appears to suit the investigation of the modularity of molecular activities [7].

More precisely, the paper is focused on three aspects of the games network: the specification of a games network, the determination of the *games network equilibria* and the reorganization of games networks. Nash equilibria are often used to characterize steady states. The games networks will be used to study steady states of a system while considering the conditions of passage of the local stable states toward a global state. Especially, from the conditions, it is possible for us to establish structural equivalences between various networks; equivalences which are based on the equality of the steady states. To a certain extent, the equality may be considered as an equivalence between dynamics. The specifications of a variety of dynamically equivalent structures puts forward the problem to characterize other comparison criteria of structural nature. Indeed, it enables us to reorganize a games network in order to have structural variant of an initial games network. Reorganizing a games network provides structurally different models, but which have the same dynamics. Scanning different models helps to get different standpoints on a studied system which puts the emphasize on different features of the system. For example, in biology reorganizing a games network is performed to discover groups of biological agents whose regulation tightly depends on other agents of the group. It is often expected that such agents play a significant role in the regulation [2, 4].

Generally speaking, reorganizing a games network while preserving the dynamics, stresses the different features of the system which support the observed dynamics. A particular attention is paid to the more detailed games networks, that is, games networks which cannot be reorganized according to the principles given for the reorganization. To some extent, they constitute the basic building blocks of the considered games network.

[1] English translation of *réseau de jeux*

In this paper, after the presentation of the games network theory, we propose an automatic method to reorganize a games network in order to get fully detailed games networks. Their computation corresponds to the computation of a normal form according a set of allowed transformations.

The paper is organised as follows : Section 2 briefly recalls the main results on strategic game theory. It specifies also certain definitions adapted to the context of the paper. Section 3 presents the extension of strategic games to games networks. We detail the games network structure. Section 4 deals with the combination of games in a games network. We define the central notion of *games network equilibrium* in a game. From this notion we introduce the notion of equivalence between games networks. This equivalence is based on the preservation of the equilibria between games networks. From this condition, we define an automatic procedure to reorganize games networks. It aims at characterizing the normal form of a games network, introduced in section 5. The reorganization is based on a separation process of the games network. In section 6, we discuss the contribution of the theory of the games networks for the analysis of the modular behaviors. We illustrate the way in which the games networks models the effects of the coupling of under systems. We conclude in section 7. Throughout the paper, examples and applications are mostly inherited from molecular biology where modules appear to be a central notion for explanation of biological functions, but games network theory can be applied to other fields which require a modular specification of interactions.

1.1 Notations

In the paper, we use the following notations:

- $[a : b] = \{i \in \mathbb{Z} | a \leq i \leq b\}$ denotes a discrete interval bounded by a and b.
- $|A|$ denotes the cardinal of a set A.
- Let $i \in A$, i also denotes the singleton $\{i\}$ if it is required by the context of the operation.
- nth denotes the rank of an integer in an integer subset according to the natural order

$$\mathsf{nth} : \mathbb{N}^+ \times \mathbf{2}^{\mathbb{N}^+} \mapsto \mathbb{N}^+, \mathsf{nth}(i, A) = \begin{cases} |\{j \in A \mid j \leq i\}| & \text{if } i \in A \\ \text{undefined} & \text{else} \end{cases}$$

  For example, we have $\mathsf{nth}(5, \{1, 3, 5, 7\}) = 3$.
- Let $C = \{C_i\}_{i \in A}$ be a set of sets, we note $C_A = \times_{i \in A} C_i$.
- Let $A = [1 : n]$, given $C = \{C_i\}_{i \in A}$, we denote by $C^*_A = \bigcup_{X \subseteq A} C_X$, the set of all $k-$uples of C with $0 \leq k \leq n$
- We consider the lifted version of A and noted $A_{\mathsf{lift}} = A + \{\bot, \top\}$ where the elements *Bottom* denoted by $\bot$ and *Top* denoted by $\top$ are added to A.
- We define the operator $\oplus : A_{\mathsf{lift}} \times A_{\mathsf{lift}} \mapsto A_{\mathsf{lift}}$ as follows:
  - $\forall \alpha \in A_{\mathsf{lift}}, \alpha \oplus \bot = \bot \oplus \alpha = \alpha$,
  - $\forall \alpha \in A_{\mathsf{lift}}, \alpha \oplus \top = \top \oplus \alpha = \top$,
  - $\forall \alpha \in A_{\mathsf{lift}}, \alpha \oplus \alpha = \alpha$,
  - $\forall (\alpha, \alpha') \in A^2_{\mathsf{lift}}, \alpha \neq \alpha' \Rightarrow \alpha \oplus \alpha' = \top$.

  $(A_{\mathsf{lift}}, \oplus)$ is a semi-group with $\bot$ as a neutral element and $\top$ as an absorbing element.

– The extension of the operator to profiles (or vectors) and set of profiles is defined as follows, considering $C = \{C_i\}_{i \in A}$, a set of sets:

$$\forall (c, c') \in C_A^2, c \oplus c' = (c_i \oplus c'_i)_{i \in 1:n}$$

$$\forall (C, C') \in 2^{C_A}, C \oplus C' = \{c \oplus c' | c \in C, c' \in C'\}$$

For example, we have:

$$(a, b, c) \oplus (a, d, \bot) = (a, \top, c),$$
$$\{(a, b, c), (a, d, c)\} \oplus \{(a, d, \bot), (x, y, c)\} = \{(a, \top, c), (a, d, c), (\top, \top, c)\}$$

– Let $X \subseteq A^n_{\mathsf{lift}}, n \geq 1$, we denote by $\lceil X \rceil$ the set of profiles of X where each profile does not contain neither $\bot$ nor $\top$, that is,

$$\lceil X \rceil = \{c \in X | \forall i \in [1:n], c_i \neq \top \wedge c_i \neq \bot\}$$

Concerning the profiles or vectors, we adopt the following notations: given $A = [1:n]$, given a profile $c \in C_A$ of a set $C_A = \times_{i \in A} C_i$, we denote by:

– $c_{-i} = (c_1, \cdots, c_{i-1}, c_{i+1}, \cdots, c_n)$; this excludes the i^{th} component of a profile.
– $(c_{-i}, \mathbf{c}_i) = (c_1, \cdots, c_{i-1}, \mathbf{c}_i, c_{i+1}, \cdots, c_n) = c$; the notation distinguishes the i^{th} component of the profile from the others. This notation is extended to sets of indices, $(c_{-X}, c_X), X \subset [1:n]$.
– Let $C = \{C_i\}_{i \in A}$; we note $C_{-i} = \times_{j \in A - i} C_j$.

We use the scatter operator which scatters values of a profile.

Definition 1 (Scatter).
We define the Scatter operator *as follows:* $c \uparrow_A^B : C_A^* \times 2^{\mathbb{N}^+} \times 2^{\mathbb{N}^+} \mapsto (C_A^*)_\bot$,

$$\forall i \in [1:\max(B)], (c \uparrow_A^B)_i = \begin{cases} c_{\mathsf{nth}(j,A)} & \exists j \in A \cap B \text{such that } \mathsf{nth}(j, B) = i \\ \bot & \text{else} \end{cases}$$

Example 1. For instance, given the following profile (a, b, c, d), we have:

$$(a, b, c, d) \uparrow_{\{1,3,7\}}^{[1:8]} = (a, \bot, b, \bot, \bot, \bot, c, \bot)$$

$$(a, b, c, d) \uparrow_{\{1,3,7\}}^{\{1,2,3,7,8\}} = (a, \bot, b, c, \bot, \bot, \bot, \bot)$$

$$(a, b, c, d) \uparrow_{\{1,3,5,17\}}^{\{1,2,8\}} = (a, \bot, \bot, \bot, \bot, \bot, \bot, \bot, \bot)$$

2 Strategic Games

In this section we give definitions of the game theory used in this paper. Although the section is mainly devoted to a summary of the definitions of the main notions encountered in game theory, we also refine some of them. Especially, we define a dedicated structure for strategies and we define the notion of dependency between agents.

The reader may refer to the books [15, 11, 5] for a complete overview of the game theory and their applications.

2.1 Definition of a strategic game

Strategic Game is a model of interplays where each agent chooses its plan of action (or strategy) once and for all, and these choices are made simultaneously. Moreover each agent are rational and perfectly informed of the payoff function of other agents. Thus they aim at maximizing their payoffs while knowing the expectation of other agents.

Definition 2 (Normal or Strategic Representation).
A strategic Game Γ is a 3−uple $\langle A, C, u\rangle$ where:

- *A is a set of players or agents.*
- *$C = \{C_i\}_{i\in A}$ is a set of strategy sets where each C_i is a set of strategies available for the agent i, $C_i = \{c_i^1, \cdots, c_i^m\}$.*
- *$u = (u_i), i \in A$ is a vector of functions where each $u_i : C \mapsto \mathbb{R}, i \in A$ is the payoff function of the agent i.*

Usually, the number of agents as well as the number of strategies are supposed to be greater than 2. However, in the paper, we admit games having either one strategy or one agent. We said that such games are *degenerated* because they do not represent strategic interactions between agents. But in our model, they remain games because the game structure can be defined and the equilibrium can be computed.

In order to conveniently combine sets of strategies, we define the strategy as follows:

Definition 3 (Set of Strategies). *Let $\langle A, C, u\rangle$ be a strategic game, let Σ^* be a set of labels. The set of strategies $C = \{C_i\}_{i\in A}$ is defined as follows $\forall i \in A, C_i = \{(i, \sigma), \sigma \in \Sigma^*\}$.*

By this definition, the fact that agents share the same strategies do not interfere in the union of sets of strategies. It fulfills the following property (proposition 1) which is important for the union of strategies.

Proposition 1. *Let A, A' be two sets of agents,*

$$\forall C_A, \forall C_{A'}, A \cap A' = \emptyset \Rightarrow C_A \cup C_{A'} = C_A \dot{\cup} C_{A'}$$

(recall that $\dot{\cup}$ stands for the disjoint union)

From given properties of the payoff functions like dominance, it is possible to characterize properties on game equilibria. The property which will interest us in this paper is the *Pareto optimality.* This property will be used in proposition 2.

Definition 4 (Pareto optimality of a strategy).
Let $\langle A, C, u\rangle$ be a strategic game, the strategy $c_i \in C_i$[2] is Pareto optimal *if*

$$\forall c_{-i} \in C_{-i}, \forall c'_i \in C_i, \forall j \in A, u_j(c_{-i}, c_i) \geq u_j(c_{-i}, c'_i)$$

For games with two players, the structure can be conveniently represented by a tableau t. The cell $t_{i,j}$ contains the pair $(u_1(c_1^i, c_2^j), u_2(c_1^i, c_2^j))$. The definition 5 details the representation by tableau.

[2] Note that c_i does not represent the i^{th} strategy of an agent, but a strategy of the i^{th} agent

Definition 5 (Representation of a $2 \times 2-$Game by a Tableau).
Given a $2 \times 2-$game $\left\langle \{1,2\}, \left((c_1^1, c_1^2), (c_2^1, c_2^2)\right), (u_1, u_2)\right\rangle$, such that the payoff values are :

$$\begin{array}{ll} u_1(c_1^1, c_2^1) = w_1 & u_2(c_1^1, c_2^1) = w_2 \\ u_1(c_1^1, c_2^2) = x_1 & u_2(c_1^1, c_2^2) = x_2 \\ u_1(c_1^2, c_2^1) = y_1 & u_2(c_1^2, c_2^1) = y_2 \\ u_1(c_1^2, c_2^2) = z_1 & u_2(c_1^2, c_2^2) = z_2 \end{array}$$

Then the tableau is defined as follows :

	c_2^1	c_2^2
c_1^1	(w_1, w_2)	(x_1, x_2)
c_1^2	(y_1, y_2)	(z_1, z_2)

Example 2. For instance, let us consider the following game, $G = \langle A, C, u\rangle$. Strategies describe characteristics states of the agents. In the system, we consider that the agent a_1, a_2 have two characteristic states $\{.\text{F}., .\text{T}.\}$ where .F. stands for false or off and .T. stands for true or on. The game works as a switch between the two agents – that is, if an agent is .T., the other one is .F.and conversely. The payoff function maximizes the .T.state. The normal form is defined as follows:

- $A = \{a_1, a_2\}$
- $C = \{\{.\text{F}._1, .\text{T}._1\}, \{.\text{F}._2, .\text{T}._2\}\}$. In the sequel, we omit the index $1, 2$ for the strategies if no ambiguity occurs.
- $u =$

	.F.	.T.
.F.	$(0,0)$	$(1,1)$
.T.	$(1,1)$	$(0,0)$

To precisely describe the interplays occurring in a game, we define the notion of *dependency* between agents. Informally, an agent is dependent on another if its choices are altered by the strategies of the other player.

Definition 6 (Agent dependency).
Let $\langle A, C, u\rangle$ be a strategic game, let $j, i \in A^2, i \neq j$ be two agents, j is said to be dependent *on i for the choice of its strategies, denoted by $i\delta_u j$, if:*

$$\exists c_i \in C_i, \exists c'_i \in C_i, \exists c_{-i} \in C_{-i}, u_j(c_{-i}, c_i) \neq u_j(c_{-i}, c'_i)$$

By extension a *strategic game* is dependent if it exists at least a dependency between two agents of the game. A game is *strongly dependent* if any agent of the game is dependent on another one. The dependencies provide an overview of the interplays of the agents in a game without having carefully studied the payoff function. To get an abstraction of the dependencies according to a game, we introduce a new representation named *the agent dependence graph*.

Definition 7 (Agent Dependence Graph).
Let $G = \langle A, C, u\rangle$ be a strategic game, the agent dependence graph $D_G = \langle A, E\rangle$ is a graph such that: $E = \{(i, j) | i\delta_u j\}$

Definition 8 (Set of predecessors).
Let $G = \langle A, C, u \rangle$ be a strategic game. We denote by $\delta_u^-(j)$, $j \in A$, the set of predecessors of j in the dependence graph of game G, that is

$$\forall j \in A, \delta_u^-(j) = \{i \in A | i\delta_u j\}$$

Example 3. In the previous game introduced in example 2, the two agents are mutually dependent $(2\delta_u 1, 1\delta_u 2)$ because:

$$u_1(.\text{T}., .\text{F}.) \neq u_1(.\text{T}., .\text{T}.) \text{ and } u_2(.\text{F}., .\text{T}.) \neq u_2(.\text{T}., .\text{T}.)$$

2.2 Nash Equilibrium

Nash Equilibrium is the central concept of the game theory. This notion captures the steady states of the play of a strategic game in which each agent holds the rational expectation about the other players behavior. *Pure Nash Equilibrium* (PNE) corresponds to a *strategic profile* c (or vector) where c_i is the strategy "chosen" by the player i. A pure Nash equilibrium is defined as follows:

Definition 9 ((Pure) Nash Equilibrium of a Strategic Game).
Let $\langle A, C, u \rangle$ be a strategic game, a Pure Nash Equilibrium *is a profile $c^* \in C_A$ of strategies with the property that :*

$$\forall i \in A, \forall c_i \in C_i, u_i(c^*_{-i}, c_i) \leq u_i(c^*_{-i}, c^*_i)$$

In other words, *no agent can unilaterally deviate from a PNE without decreasing its payoff.*

A pure Nash equilibrium can be considered as *a best reply* for all the agents, that is the reply which maximizes the outcome while considering the strategies of the other agents. We define the set of the pure Nash equilibria of a game as follows (definition 10):

Definition 10 (Set of Pure Nash Equilibria).
Let $G = \langle A, C, u \rangle$ be a game, we define the set of pure Nash equilibria:

$$\mathsf{pne}(G) = \{c^* \in C_A | u_i(c^*_{-i}, c_i) \leq u_i(c^*_{-i}, c^*_i), \forall i \in A, \forall c_i \in C_i\}$$

Pure Nash Equilibria are extended to Mixed Nash Equilibria which is based on a probabilistic definition of the strategic profile c where each c_i becomes a vector of probabilities, representing the distribution on player's strategies [12, 11].

3 Games Network

A games network corresponds to an extension of the game theory which defines *modular interactions* localized to different subsets of agents. Each module corresponds to a specific game defined by a payoff function. Parameters of the payoff function are strategies of agents involved in the game. Agents are shared between different modules and play different games in parallel. However they have the same set of strategies for every games they play. In a games network, several games are combined to form a more general structure of Networks. In this section, we address the main definitions of a games network.

3.1 Definition of a Games Network

The definition of a games network mainly consists in defining a set of agents connected to a set of games. The normal form of a games network is as follows:

Definition 11 (Games Network).
A games network is a $3-$*uple* $\langle \mathcal{A}, C, \mathcal{U} \rangle$ *where*

- $\mathcal{A}$ *is a set of agents or players.*
- $C = \{C_i\}_{i \in \mathcal{A}}$ *is a set strategies.*
- $\mathcal{U} = \{\langle A, u \rangle\}$ *is a set of* game nodes *where each* $A \subseteq \mathcal{A}$ *is a set of agents and* $u : A \times C_A \mapsto \mathbb{R}$ *is a set of payoff functions such that* $u = \{u_i : C_A \mapsto \mathbb{R}\}_{i \in A}$. *(We can notice that, by convention, the first argument corresponding to the signature of* u *is the index* i*.)*

A games network offers a synthetic representation to define the different interplays between several players. The structure $\langle A, u \rangle$ totally determines a game played by a subset of agents since it useless to include the strategies which are the same for any agent of the network. A games network is represented by a bipartite graph $\langle \mathcal{A}, \mathcal{U}, E \rangle, E \subseteq \mathcal{A} \times \mathcal{U}$ where an edge $(i, \langle A, u \rangle)$ is a member of E if and only if $i \in A$.

The definition of the game networks would lead to the construction of some undesirable games networks. For instance, you may include degenerated game nodes with no strategic interplays (e.g. $u_i(c) = \text{cste}, \forall c, \forall i$). Thus, among possible games networks, we distinguish *well-formed games networks* to the others.

Definition 12 (Well-Formed Games Network).
Let Γ *be a games network,* Γ *is* well-formed *if:*

- *Any game node have dependent agents.*
- *A the set of agents of a game node is not a subset of any set of agents of another game nodes.*

The dependence relation introduced in section 2 for a game is extended to the dependence relation by considering a games network as follows:

Definition 13 (Dependence relation according to a games network).
Let $\Gamma = \langle \mathcal{A}, C, \mathcal{U} \rangle$ *be a games network, let* $i \in \mathcal{A}$ *and* $j \in \mathcal{A}$ *be two agents,*

$$i \delta_{\mathcal{U}} j \text{ iff } \exists G = \langle A, u \rangle \in \mathcal{U}, i \delta_u j$$

(Definitions of dependence graph and set of predecessors are extended to games networks in the same way)

3.2 Restriction

A game node can be viewed as a sub game of a larger game played by the whole agents of the network. In this section, we equip the theory with the *restriction operator* to provide the ability to put the focus on an arbitrary sub game.

Definition 14 (Profile Restriction).

Let $\mathcal{A} = [1:n]$ be a discrete interval representing a set of agents, let $\{C_i\}_{i \in \mathcal{A}}$ be a set of strategy sets, given a profile $c \in C_{\mathcal{A}}$ (recall that $C_{\mathcal{A}} = \times_{i \in \mathcal{A}} C_i$), we define its restriction to a subset $A \subseteq \mathcal{A}$, denoted by $c{\downarrow_A}\colon C_{\mathcal{A}} \times 2^A \mapsto (C_{\mathcal{A}})_{\mathsf{lift}}$, as follows[3] :

$$(c{\downarrow_A})_i = \begin{cases} c_i & \text{if } i \in A \\ \perp & \text{else} \end{cases}$$

The restriction is obviously extended to a set of profiles by applying the operation to every element.

Example 4. Let $\mathcal{A} = [1:4]$, we have $(c_1, c_2, c_3, c_4){\downarrow_{\{1,3\}}} = (c_1, \perp, c_3, \perp)$

The previous definition (14) restricts the profile to relevant values according to a subset of agents, named its *support*. A profile of values defined by a restriction is considered as a *local profile* of a subset of agents. Whatever the values associated to other agents are, they will not be considered for a local profile.

//The extension of the restriction corresponds to the scatter operation defined as follows:

By extension, given a game $G = \langle A, C, u \rangle$, the restriction of a game to a subset $X \subseteq A$, $G{\downarrow_X}$, corresponds to a game where only strategies and payoff functions of the agents of X are relevant.

Definition 15 (Game Restriction).

Let $G = \langle A, C, u \rangle$ be a strategic game, we define its restriction by a set $X \subseteq A$ as follows:

$$G{\downarrow_X} = \langle A, \{C_i\}_{i \in X} \cup \{\{\perp_j\}\}_{j \in A-X}, u_\perp \rangle$$

where

$$\forall i \in A, \forall c_X \in C_X, u_{\perp i}(\perp_{-X}, c_X) = \max\{u_i(c_{-X}, c_X) | c_{-X} \in C_{-X}\}$$

The restriction also concerns the computation of the equilibria. In this case, the computation of Nash equilibria only considers the elements specified in the restriction

Definition 16 (pne Restriction).

Let $G = \langle A, C, u \rangle$ be a game, let $X \subseteq A$ be a set of agents, we define the restriction of the Nash equilibrium property as follows:

$$\mathsf{pne}{\downarrow_X}(G) = \{c^* \in C_A | u_i(c^*_{-i}, c_i) \leq u_i(c^*_{-i}, c^*_i), \forall i \in X, \forall c_i \in C_i\}$$

Example 5. Let us consider the following game $G = \langle A, C, u \rangle$ defined by its normal form: $A = \{a_1, a_2\}$, $C = \{\{.\mathrm{F}.{}_1, .\mathrm{T}.{}_1\}, \{.\mathrm{F}.{}_2, .\mathrm{T}.{}_2\}\}$, and u the payoffs function.

$u =$

	.F.	.T.
.F.	(0, 0)	(1, 1)
.T.	(1, 0)	(0, 0)

Thus, $\mathsf{pne}(G) = \{(.\mathrm{F}.{}_1, .\mathrm{T}.{}_2), (.\mathrm{T}.{}_1, .\mathrm{F}.{}_2)\}$.

According to the previous restriction definitions, we have the following results:

[3] $\perp$ stands for an irrelevant value

- Profile Restriction:
  - $\mathsf{pne}(G)\downarrow_{\{a_2\}} = \{(\bot_1, .\text{T}._2), (\bot_1, .\text{F}._2)\}$
- Game Restriction:
  - The restriction $G\downarrow_{\{a_2\}}$ corresponds to the payoff functions :

$$u_\bot = \begin{array}{c|cc} & .\text{F}._2 & .\text{T}._2 \\ \hline \bot_1 & (1,0) & (1,1) \end{array}$$

  - $\mathsf{pne}(G\downarrow_{\{a_2\}}) = \{(\bot_1, .\text{F}._2)\}$
- pne Restriction:
  - $pne\downarrow_{\{a_2\}}(G) = \{(.\text{F}._1, .\text{T}._2), (.\text{T}._1, .\text{F}._2), (.\text{T}._1, .\text{T}._2)\}$
  - $pne\downarrow_{\{a_1\}}(G) = \{(.\text{F}._1, .\text{T}._2), (.\text{T}._1, .\text{F}._2)\}$

The restriction applied to profiles will be used in the next section to put the focus on a sub-part of a profile which corresponds to a games node. The restriction applied to games or Nash equilibria calculation can be used to reduce the computational. The preceding definition directs the reduction towards a reduction in the number of functions to be analyzed (definition 16). Concerning the calculation of the Nash equilibria, the following proposition investigates another aspect: enabling the commutation of the restriction and equilibria calculation.

Proposition 2 (*PNE* & Restriction Commutation). *Let $G = \langle A, C, u\rangle$ be a strategic game, let $X \subseteq A$ be a set of agents, if* $\forall c^* \in \mathsf{pne}(G), \forall i \in A - X, c_i^*$ is Pareto optimal *then*

$$\mathsf{pne}(G)\downarrow_X = \mathsf{pne}(G\downarrow_X)$$

Proof. To prove the commutation, it suffices to prove that $\forall c^* \in \mathsf{pne}(G), c^*\downarrow_X \in \mathsf{pne}(G\downarrow_X)$. Without loss of generality we assume that $X = A - i, i \in A$ – that is, only one agent is restricted. Assuming that c_i^* is Pareto optimal for all Nash equilibria $c^* \in \mathsf{pne}(G)$, let $j \in A$ be an agent
By definition of the Nash equilibrium, we have:

$$u_i(c_{-i}^*, c_i^*) \geq u_i(c_{-i}^*, c_i), \forall c_i \in C_i.$$

Since c_i^* is a Pareto optimal strategy by hypothesis, we have:

$$u_j(c_{-i}^*, c_i^*) \geq u_j(c_{-i}^*, c_i), \forall j \in A, \forall c_i \in C_i.$$

By definition of the Nash equilibrium and the Pareto optimal strategy, we have:

$$u_j(c_{-i-j}^*, c_i^*, c_j^*) \geq u_j(c_{-i-j}^*, c_i^*, c_j) \geq u_j(c_{-i-j}^*, c_i, c_j), \forall j \in A, \forall c_i \in C_i, \forall c_j \in C_j.$$

By definition 14 of the restriction on games, and according to the previous inequalities, we can conclude that:

$$u_j^\bot(c_{-i-j}^*, \bot_i, c_j) = u_j(c_{-i-j}^*, c_i^*, c_j), \forall j \in A, \forall c_j \in C_j.$$

Thus, the following inequality holds:

$$u_j(c_{-i-j}^*, \bot_i, c_j^*) \geq u_j(c_{-i-j}^*, \bot_i, c_j), \forall j \in A, \forall c_j \in C_j.$$

which is the condition of a Nash equilibrium. So, we have:

$$(c_{-i}^*, \bot_i) = c^*\downarrow_{A-i} \in \mathsf{pne}(G\downarrow_{A-i}).$$

■

Example 6. Let's consider two variants of the game of example 2 where the payoff functions are defined as follows :

$$① \; u = \begin{array}{c|cc} & .\text{F}. & .\text{T}. \\ \hline .\text{F}. & (2,1) & (1,0) \\ .\text{T}. & (1,0) & (0,0) \end{array}, \; u = \begin{array}{c|cc} & .\text{F}. & .\text{T}. \\ \hline .\text{F}. & (2,1) & (1,0) \\ .\text{T}. & (1,0) & (0,2) \end{array} \; ②$$

The restriction $G{\downarrow_{\{a_2\}}}$ respectively corresponds to the payoff functions :

$$① \; u_\perp = \begin{array}{c|cc} & .\text{F}. & .\text{T}. \\ \hline \perp & (2,1) & (1,0) \end{array}, \; u_\perp = \begin{array}{c|cc} & .\text{F}. & .\text{T}. \\ \hline \perp & (2,1) & (1,2) \end{array} \; ②$$

In both cases, the Nash equilibrium is $(.\text{F}., .\text{F}.)$ and its restriction according to a_2 is $(\perp, .\text{F}.)$. In case ①, the restricted Nash equilibrium of the game is also a Nash equilibrium of the restricted game. In case ②, the Nash equilibrium for the restricted game is $(\perp, .\text{T}.)$ and is different of the restriction of the Nash equilibrium of the original game. In case ①, the strategy $.\text{F}._1$ is Pareto optimal whereas there is no Pareto optimality in case ②.

We extend the restriction operator by removing bottom elements ($\perp$) from the profile, but the order of the other values is conserved in the resulting profile. We note the composition of the removals and restriction operation as follows: $c \Downarrow_X$

The following proposition gives some equivalences using the restriction operator.

Proposition 3. *Let $\langle A, C, u\rangle$ be a game, $\forall X \subseteq A$ the following propositions hold:*

$$\begin{array}{rl} c{\downarrow_X} = (c \Downarrow_X) \uparrow_X^A & , \forall c \in C_A \\ \lceil C_A{\downarrow_X} \oplus S\rceil = S & , \forall S \subseteq C_A \\ \lceil \mathsf{pne}{\downarrow_{X_1}} (G) \oplus \mathsf{pne}{\downarrow_{X_2}} (G)\rceil = \mathsf{pne}{\downarrow_{X_1 \cup X_2}} (G) & , \forall X_1 \subseteq A, \forall X_2 \subseteq A \end{array}$$

Proof. *The proves are let to the reader.* ■

Example 7. Let us consider the game defined in example 5. We have:

$$pne{\downarrow_{\{a_1\}}} (G) \oplus pne{\downarrow_{\{a_2\}}} (G) = \{(\perp_1, \perp_2), (.\text{T}._1, .\text{F}._2), (.\text{T}._1, \perp_2), (.\text{F}._1, .\text{T}._2), (\perp_1, .\text{T}._2)\}$$

Thus, $\lceil pne{\downarrow_{\{a_1\}}} (G) \oplus pne{\downarrow_{\{a_2\}}} (G)\rceil = \{(.\text{F}._1, .\text{T}._2), (.\text{T}._1, .\text{F}._2)\} = \mathsf{pne}(G)$.

3.3 Orientation

To complete the representation, we graphically (see example 8) distinguish three categories of interactions : IN, OUT and UNDIRECTED. The interaction categories correspond to labels of edges in the bipartite graphs. They are graphically represented by arrows for IN and OUT interactions and unoriented edges for UNDIRECTED interactions. They are used to qualify the payoff functions of a game node. The IN interaction means that the game node (local payoff function) does not affect the choice of the input agent, that is, given a strategy, the outcome remains the same whatever the other agents' strategies. The OUT interaction means that the strategies are the result of the game, that is the agent is dependent on the other agents but they are not dependent on it. Formally we define :

Definition 17 (Orientation of Interactions).

Let $\langle A, u\rangle$ be a game node, let $i \in A$ be an agent.

- *The interaction $(i, \langle A, u\rangle)$ is* IN *if:* $\forall c \in C_A, \forall c' \in C_A, u_i(c) = u_i(c')$
- *The interaction $(i, \langle A, u\rangle)$ is* OUT *if:* $\exists c_i \in C_i, \exists c_{-i} \in C_{-i}, \exists c'_{-i} \in C_{-i}, u_i(c_{-i}, c_i) \neq u_i(c'_{-i}, c_i)\wedge$ $\forall j \in A - i, \forall c_{-i} \in C_{-i}, \forall c_i \in C_i, \forall c'_i \in C_i, u_j(c_{-i}, c_i) = u_j(c_{-i}, c'_i)$

Given a game $G = \langle A, C, u\rangle$ we note by $\mathsf{in}(A)$ (resp. $\mathsf{out}(A)$) the set of agents having IN (resp. OUT) interactions. A game with IN interactions do not perform selections. The orientation of the graph describes the way that agents interact in a game. The following proposition simplifies the computation of the Pure Nash Equilibria because it restricts the computation to equilibria for agents which are member of the support since the chosen strategy of agents which are not members of the support is only $\perp$ by definition.

Proposition 4. *The Nash equilibria of a node game $G = \langle A, u\rangle$ having agents with* IN *interactions is defined as follows :*

$$\mathsf{pne}(G) = \mathsf{pne}{\downarrow_{A-\mathit{in}(A)}} (G)$$

Proof. : the proof is let to the reader.

Corollary 1. *The computation of Pure Nash Equilibria of a game $G = \langle A, u\rangle$ having exclusively* IN *or* OUT *interactions is determined as follows :*

$$\mathsf{pne}(G) = \mathsf{pne}{\downarrow_{\mathsf{out}(A)}} (G)$$

If we consider that a games network is the modeling of regulatory interplays between agents then IN interactions means that the agent acts as regulators since payoffs of other agent potentially depends on her strategies but it is insensitive to strategies of others by this node game. OUT interaction means that the agent is regulated by a game and it is the result of the game. In summary, setting an interaction IN or OUT to a node game is used to describe some properties of the payoff functions in order to simplify the computation of the Nash equilibrium. IN-OUT games are used to represent interplays which are mainly directed by some agents which governs the definition of the equilibria. They are considered to be “easy” because the computation of the pure nash equilibria relies on the computation of the maximal outcomes of the OUT agent (corollary 1)

This can be used to describe a behavior of an agent which can be modeled by a function taking strategies of other as arguments.

Given a function $f : C_{-i} \mapsto C_i$, the pure Nash Equilibria of following payoff function u_f gives the extensive definition of the function in the form $((c_{-i}, f(c_{-i})))$– that is every profiles $(c_{-i}, f(c_{-i}))$ belong to a pure Nash Equilibria.

Proposition 5. *Let $f : C_X \mapsto C_i$ be a function, let $G = \langle A, C, u\rangle$ be a strategic game such that $A = X \cup \{i\}$ (note that $C_{-i} = C_X$) where u is defined as follows :*
let $(\alpha_0, \alpha, \beta) \in \mathbb{R}^3$ such that $\alpha < \beta$, we have

$$u_i(c) = \begin{cases} \beta \text{ if } c = (c_{-i}, f(c_{-i})) \\ \alpha \text{ otherwise} \end{cases}$$
$$u_j(c) = \alpha_0, \forall j \neq i$$

We have : $\mathsf{pne}(G) = \{(c_{-i}, f(c_{-i}))\ |c_{-i} \in C_X\}$

Proof. *the proof is let to the reader.* ■

Example 8. The following games network

$$\langle\{a_1, a_2, a_3\}, \{\{.\text{F}., .\text{T}.\}, \{.\text{F}., .\text{T}.\}, \{.\text{F}., .\text{T}.\}\}, \{\langle\{a_1, a_2, a_3\}, u\rangle\rangle$$

describes the boolean OR operation – that is, $a_3 = a_1 \vee a_2$. The table u below describes the payoff functions u.

a_1	a_2	a_3	u_1	u_2	u_3
.F.	.F.	.F.	0	0	1
.F.	.F.	.T.	0	0	0
.F.	.T.	.F.	0	0	0
.F.	.T.	.T.	0	0	1
.T.	.F.	.F.	0	0	0
.T.	.F.	.T.	0	0	1
.T.	.T.	.F.	0	0	0
.T.	.T.	.T.	0	0	1

According to the table, we can conclude that $\mathsf{in}(G) = \{a_1, a_2\}$ and $\mathsf{out}(G) = \{a_3\}$ which can be represented by the following games network.

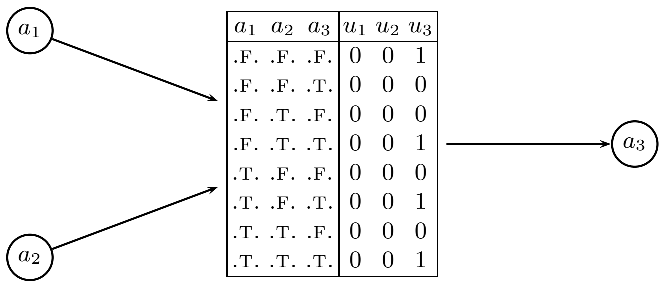


Fig. 1.

This can also be more concisely described by : $u_1(c) = 0, u_2(c) = 0, u_3(c) =$ if $(c_1 \vee c_2 = c_3)$then 1 else $0, \forall c \in \{.\text{F}., .\text{T}.\}^3$

4 Combination of Games

The definition of a games network allows the combination of several games into a single network. This puts the emphasis on the way that the network structure is determined, because different structures can be proposed to model the same situations. In order to compare them, it is necessary to identify the equivalence between games networks. The conditions of equivalence investigated in the paper are based on the equilibria. Informally, two games are equivalent if their equilibria are the same. Such a condition requires to enlarge the equilibrium locally computed from game nodes to the whole games network.

The equilibrium at the scale of the network is named the *games network equilibria* (Gne). The Gne extends local equilibrium to subnetworks. The subsection 4.1 formally defines the equilibria.

4.1 Games Network Equilibrium

An extension of the Nash equilibrium to networks consists in defining an equilibrium at the scale of the network. Informally, a games network equilibrium corresponds to a compatible association of local equilibria. We assume that agents follow the *single played strategy* rule, that an agent plays the same strategy for every connected games. Hence, a global network equilibria must respect this rule. The definition of Gne can of course be applied to the whole network. But the restriction to the a subset of game node allow us to define regions where equilibria are compatible.

Definition 18 (Pure Games Network Equilibrium).
Let $\Gamma = \langle \mathcal{A}, C, \mathcal{U} \rangle$ be a games network, let $c^* = (c_1, \cdots, c_n)$ be a strategy profile of every agents, (recall that by convention $|\mathcal{A}| = n$), c^* is a pure games network equilibrium of a subset $U \subseteq \mathcal{U}$ if:
$\forall \langle A, u \rangle \in U, c^* \Downarrow_A$ is a pure Nash equilibrium of the game $\langle A, (C_i)_{i \in A}, u \rangle$.

The definition of the Gne provides the ability to define an equivalence between different representations of a games network. An equivalence between two games networks is based on the equality of their equilibria. Informally, it means that both games networks have the same dynamics if we admit that equilibria represent steady states. However, the equivalence between two structures is based on the largest set of global sets of equilibria which is defined as follows:

Theorem 1 (Largest Set of Global Equilibria).
Let $\Gamma = \langle \mathcal{A}, C, \mathcal{U} \rangle$ be a games network, let $U \subset \mathcal{U}, U = \{g_i = \langle A_i, u_i \rangle\}$ be a set of game nodes, and let $A = \bigcup_i A_i$. Then, $\mathsf{Gne}_\Gamma(U) = \left\lceil \bigoplus_i \mathsf{pne}(g_i) \uparrow_{A_i}^{A} \right\rceil$ is the largest set of games network equilibria $c^* \in C_A$ for games nodes of U

Proof.
- *Assume that it exists a games network equilibrium for U, $c^* \in C_{\cup_i A_i}$.*
- *By definition 18 of pure games network equilibrium we have, $\forall \langle A, u \rangle \in U, c^* \Downarrow_A$ is a local Nash equilibrium.*
- *By definition of restriction operator, c^* can be rewritten as $c^* = \bigoplus_i c^* \downarrow_{A_i}$.*
- *According to the proposition 3, we have $c^* = \bigoplus_i c^* \Downarrow_{A_i} \uparrow_{A_i}^{\mathcal{A}}$ with $c^* \Downarrow_{A_i} \in \mathsf{pne}(g_i)$*
- *So c^* belongs to $\mathsf{Gne}_\Gamma(U)$.*

■

In the sequel, we omit Γ if no ambiguity occurs.

Definition 19 (Gne Equivalence).
Let $\Gamma_1 = \langle \mathcal{A}_1, C_1, \mathcal{U}_1 \rangle$ and $\Gamma_2 = \langle \mathcal{A}_2, C_2, \mathcal{U}_2 \rangle$ be two games networks such that $\mathcal{A}_1 = \mathcal{A}_2, C_1 = C_2$.
Γ_1 and Γ_2 are equivalent, denoted by $\Gamma_1 \equiv_{\mathsf{Gne}} \Gamma_2$, if and only if $\mathsf{Gne}_{\Gamma 1}(\mathcal{U}_1) = \mathsf{Gne}_{\Gamma 2}(\mathcal{U}_2)$

4.2 Conditions of separation

The previous requirement defines the structural equivalences according to the equivalence of the dynamics. This opens on the possibility of establishing transformations of a structure. We will see that this possibility reveals the importance of the observer when setting agents in module. In this subsection we focus on the basic operations operating on games nodes, that is, joining and separating games.

Restructuring games networks will be expressed in terms of substituting game nodes by others. Definition 20 formally defines this operation.

Definition 20 (Substitution).
Let $\Gamma = \langle \mathcal{A}, C, \mathcal{U} \rangle$ be a games network, let $U = \{g_i = \langle A_i, u^i \rangle\}, U \subseteq \mathcal{U}$ be a set of games nodes, let $U' = \{\langle A_{i'}, u^{i'} \rangle\}$ be another set of game nodes such that $\forall i', A_{i'} \subseteq \mathcal{A}$, we define the substitution, denoted by $\Gamma_{[U/U']}$ as follows:.

$$\Gamma_{[U/U']} = \langle \mathcal{A}, C, \mathcal{U} - U \cup U' \rangle$$

We define the Join operation between game nodes as follows:

Definition 21 (Join according to ω).
Let $\Gamma = \langle \mathcal{A}, C, \mathcal{U} \rangle$ be a games network, let $G_1 = \langle A_1, u^1 \rangle$ and $G_2 = \langle A_2, u^2 \rangle$ be two game nodes of Γ ($G_1 \in \mathcal{U}, G_2 \in \mathcal{U}$), let $\omega : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ be a function, we define: $G_1 \bigvee^{\omega} G_2 = \langle A_1 \cup A_2, u \rangle$ with :

$$\begin{aligned} \forall c \in C_{(A_1 \cup A_2)}, \; & \forall i \in A_1 - A_2 \; u_i(c) = {u^1}_i(c \Downarrow_{A_1}) \\ & \forall i \in A_2 - A_1 \; u_i(c) = {u^2}_i(c \Downarrow_{A_2}) \\ & \forall i \in A_1 \cap A_2 \; u_i(c) = \omega({u^1}_i(c \Downarrow_{A_1}), {u^2}_i(c \Downarrow_{A_2})) \end{aligned}$$

The join operation depends on a function ω. For instance, the maximum function $\max(v_1, v_2)$ can be a candidate for giving a concrete definition of $\bigvee$ operation. If no specific property on ω is required we omit it in the specification of the operation.

The join operation or, conversely, the separation are the basic operations for games networks reorganization. However, the reorganization can be performed if the initial games network and that resulting of the reorganization are equivalent in the sense of the definition 19. Definition 22 and theorem 2 address the equivalence between games networks which can be obtained by joining or separating games nodes. It provides a general condition to restructure games networks based on the preservation of the equilibria. A special attention is paid on the reciprocal operation of the join because it enables us to split a games network into another one composed of more elementary games. The separation, according to a function ω, is defined as follows:

Definition 22 (Separation according to ω).
Let $\Gamma = \langle \mathcal{A}, C, \mathcal{U} \rangle$ be a games network, a game node $G = \langle A, u \rangle \in \mathcal{U}$ is said to be separable (according to ω) if:

$$\begin{aligned} & \exists G_1 = \langle A_1, u^1 \rangle, \exists G_2 = \langle A_2, u^2 \rangle \text{ such that} \\ & \quad G_1 \textstyle\bigvee^{\omega} G_2 = G \text{ and } \mathsf{pne}(G_1 \textstyle\bigvee^{\omega} G_2) = \mathsf{Gne}(\{G_1, G_2\}) \end{aligned}$$

It leads to the following theorem which defines a basic condition to perform modifications of the network.

Theorem 2. *Let $\Gamma = \langle \mathcal{A}, C, \mathcal{U} \rangle$ be a games network, Let $G \in \mathcal{U}$ be a game node,let $G_1 = \langle A_1, u^1 \rangle, G_2 = \langle A_2, u^2 \rangle$ be two game nodes such that $A = A_1 \cup A_2$, if G is separable according to ω to G_1, G_2 then we have:*

$$\Gamma \equiv_{\mathsf{Gne}} \Gamma_{[G/\{G_1,G_2\}]}$$

Proof. The proof is an immediate consequence of the definition 22. ■

The following propositions establish some relationships between game separation and depencies. They are central to automatically decide whether or not a game is separable. Application of the propositions and network modifications will be presented in the nex section.

Proposition 6. *Let $\Gamma = \langle \mathcal{A}, C, \mathcal{U} \rangle$ be a games network, and let $G = \langle A, u \rangle \in \mathcal{U}$ be a game node wich is separable into $G_1 = \langle A_1, u^1 \rangle$ and $G_2 = \langle A_2, u^2 \rangle$. Then*

$$\forall (i,j) \in \mathcal{A}^2, i\delta_u j \Rightarrow \{i,j\} \subseteq A_1 \vee \{i,j\} \subseteq A_2$$

Proof. Let Γ, G, G_1, G_2 be defined according to the proposition, and let us suppose we have found $(i,j) \in A^2$ such that

$$i\delta_u j \wedge \{i,j\} \not\subseteq A_1 \wedge \{i,j\} \not\subseteq A_2$$

Because $G_1 \bigvee G_2 = G$, we have $A_1 \cup A_2 = A$. So considering $i \in A_1$, we have

$$i \in A_1 - A_2, j \in A_2 - A_1$$

By definition of the separation and joint:

$$\forall (c_i, c_i') \in C_i^2, \forall c_{-i} \in C_{-i} u_j(c_{-i}, c_i) = u_j^2((c_{-i}, c_i) \Downarrow_{A_2})$$

Because $i \notin A_2, \forall (c_i, c_i') \in C_i^2, \forall c_{-i} \in C_{-i}, (c_{-i}, c_i) \Downarrow_{A_2} = (c_{-i}, c_i') \Downarrow_{A_2}$. So

$$\forall (c_i, c_i') \in C_i^2, \forall c_{-i} \in C_{-i}, u_j(c_{-i}, c_i) = u_j(c_{-i}, c_i')$$

That is $i \not\delta_u j$, which is false. ■

Proposition 7. *Let $\Gamma = \langle \mathcal{A}, C, \mathcal{U} \rangle$ be a games network, and let $G = \langle A, u \rangle \in \mathcal{U}$ be a game node wich is separable according to ω into $G_1 = \langle A_1, u^1 \rangle$ and $G_2 = \langle A_2, u^2 \rangle$. Then*

$$\forall (i,j) \in \mathcal{A}^2, i\delta_u j \Rightarrow i\delta_{u^1} j \vee i\delta_{u^2} j$$

Proof. Let Γ, G, G_1, G_2 be defined according to the proposition, and let us suppose we have found $(i,j) \in A^2$ such that

$$i\delta_u j \wedge i \not\delta_{u^1} j \wedge i \not\delta_{u^2} j$$

Because of proposition 6, we know that either $\{i,j\} \subseteq A_1$ or $\{i,j\} \subseteq A_2$. Let us suppose that $\{i,j\} \subseteq A_1$. So, two cases should be considered.

– *If* $j \notin A_2$.

$$\forall (c_i, c'_i) \in C_i^2, \forall c_{-i} \in C_{-i}, u_j(c_{-i}, c_i) = u_j^1((c_{-i}, c_i) \Downarrow_{A_1})$$

Because $i \not\triangleright_{u^1} j$, $u_j^1((c_{-i}, c_i) \Downarrow_{A_1}) = u_j^1((c_{-i}, c'_i) \Downarrow_{A_1})$. *Thus,*

$$\forall (c_i, c'_i) \in C_i^2, \forall c_{-i} \in C_{-i}, u_j(c_{-i}, c_i) = u_j(c_{-i}, c'_i)$$

And $i \not\triangleright_u j$

– *If* $j \in A_2$.

$$\forall (c_i, c'_i) \in C_i^2, \forall c_{-i} \in C_{-i}, u_j(c_{-i}, c_i) = \omega(u_j^1((c_{-i}, c_i) \Downarrow_{A_1}, u_j^2((c_{-i}, c_i) \Downarrow_{A_2})$$

However, $i \not\triangleright_{u^1} j \wedge i \not\triangleright_{u^2} j$. *Thus,*

$$\forall (c_i, c'_i) \in C_i^2, \forall c_{-i} \in C_{-i}, u_j(c_{-i}, c_i) = u_j(c_{-i}, c'_i)$$

And $i \not\triangleright_u j$, *which is false.*

■

5 Games Network Normal Form

The previous section describes the necessary conditions to operate the separation of a game node. They are based on conservation of the equilibria of the games network. Separation depends on a particular function ω which embodies the standpoint of the observer on the structuring of the system.

The importance of the reorganization of a game lies in the simplification which it allows. Separation changes the interactions which seem complex at first sight into combinations of interactions implying less agents and, thus, simpler to comprehend. It helps to reduce the complexity of the description of the system without reducing that of its dynamics because the equilibria are preserved.

This puts the emphasis on the games network reorganization to simplify it. Being given a games network Γ, we aim at defining a *Games Network Normal Form* – that is, a reorganization which cannot be reorganized again according to the defined constraints. However reorganization may generate infinite alternatives of games networks from a given games network. Obviously, we have $G = G \bigvee G$ whatever the game node G. More generally, if we assume that ω selects the first argument regardless the value of the second one then given a game node $G = \langle A, u \rangle$, we have $G = G \bigvee^{\omega} G', G' = \langle A', u' \rangle$ providing $\mathsf{pne}(G) = \mathsf{pne}(G')$ and $A' \subseteq A$. Thus, without additional constraints there is no *a priori* unicity of the normal form.

Moreover, it seems also desirable that a normal form addresses a class of functions instead of a specific function because we obtain a more general process for the reorganization. Indeed, if we admit that ω formalizes the viewpoint of the observer, then, by addressing a class of the functions Ω, the reorganization is compatible with the viewpoints of all the observers of this class.

Among possible classes of functions, some of them appear to be more relevant for modeling. We address the computation of the normal form for *functions with neutral element*. It is defined as follows:

Definition 23 (Function with Neutral Element).
Let Ω be the set of idempotent function with neutral element defined as follows:

$$\Omega = \{\omega : \mathbb{R}^2 \mapsto \mathbb{R} | \exists e_\omega \in \mathbb{R}, \forall x \in \mathbb{R}, \omega(x, e_\omega) = \omega(e_\omega, x) = x\}$$

In the sequel, the neutral element will be denoted by e if we do not consider a specific function of Ω but a generic instance of them.

The subsection 5.1 gives the definition of a normal form. The subsection 5.2 defines $\Omega-$normal form, and gives an algorithm to compute it.

5.1 Normal form according to a function

The normal form is defined as follows:

Definition 24 (Normal Form according to a function).
Let Γ be a games network, $\omega : \mathbb{R}^2 \mapsto \mathbb{R}$ a function.
*Γ is said to be $\omega-$*normal *if it is inseparable according to ω.*

A normal form can be computed by successive separations, that is each sub-game of a game is obtained by separation according to the considered function ω. It relies on the identification of topological properties of the agent dependence graph. When separation is applied, the agents are distributed in the two games resulting from separation. The criteria governing separation may be determined according to the dependence graph. In this case, they result from the impact that separation has on the agents. According to definition 21, the problem is reduced to the way in which the payoff function of each games node is computed from the payoff function of the original game.

The following propositions show that a normal form obtained by separation preserves the dependences and that the agents belonging to the same game of the normal form must be in dependence relation.

Proposition 8. *Let $\Gamma = \langle \mathcal{A}, C, \mathcal{U} \rangle$ be a games network. Let $\Gamma_f = \langle \mathcal{A}, C, \mathcal{U}_f \rangle$ be a normal form of Γ, obtained by separation. Then*

$$\forall (i, j) \in \mathcal{A}^2, i\delta_{\mathcal{U}} j \Rightarrow i\delta_{\mathcal{U}_f} j$$

Proof. The proposition is an immediate consequence of propositions 6 and 7.

Proposition 9. *Let $\Gamma = \langle \mathcal{A}, C, \mathcal{U} \rangle$ be a games network, let $\Gamma_f = \langle \mathcal{A}, C, \mathcal{U}_f \rangle$ be a normal form of the games network Γ obtained by separation. Then*

$$\forall G_f = \langle A_f, u^f \rangle \in \mathcal{U}_f, \forall X_1 \subseteq A_f, \forall X_2 \subseteq A_f,$$
$$X_1 \cap X_2 = \emptyset \wedge X_1 \cup X_2 = A \Rightarrow \exists i \in X_1, \exists j \in X_2, i\delta_{\mathcal{U}} j \vee j\delta_{\mathcal{U}} i$$

Proof. Assume that

$$\exists G = \langle A, u \rangle \in \mathcal{U}_f, \exists X_1 \subseteq A, \exists X_2 \subseteq A,$$
$$X_1 \cap X_2 = \emptyset \wedge X_1 \cup X_2 = A \wedge (\forall i \in X_1, \forall j \in X_2, i \not\delta_u j \wedge j \not\delta_u i)$$

By the independence which implies the invariance of payoffs of agents of X_1 for configurations of C_{X_2} and conversely, we can define two games $G_1 = \langle X_1, u^1\rangle$ and $G_2 = \langle X_2, u^2\rangle$ as follows:

$$u^1_i(c \Downarrow_{X_1}) = u_i(c \Downarrow_{X_1}, c \Downarrow_{X_2})\ \forall c \Downarrow_{X_2} \in C_{X_2}, \forall i \in X_1$$
$$u^2_j(c \Downarrow_{X_2}) = u_j(c \Downarrow_{X_1}, c \Downarrow_{X_2})\ \forall c \Downarrow_{X_1} \in C_{X_1}, \forall j \in X_2$$

By definition of the join operator (definition 21), we have:

$$G = G_1 \bigvee G_2$$

To prove the separation we also must prove the equivalence between equilibria. By definition,

$$\mathsf{pne}(G) = \{c^* \in C_A \mid u_i(c^*) \geq u_i(c), \forall i \in A, \forall c \in C_A\}$$

$$\mathsf{Gne}(G_1, G_2) = \{c^* \in C_A \mid c^* \Downarrow_{X_1} \in \mathsf{pne}(G_1) \wedge c^* \Downarrow_{X_2} \in \mathsf{pne}(G_2)\}$$

However $\mathsf{pne}(G_1) = \{c^*_1 \in C_{X_1} \mid u^1_i(c^*_1) \geq u^1_i(c_1), \forall i \in X_1, \forall c_1 \in C_{X_1}\}$*, thus we have:*

$$\begin{aligned}\mathsf{Gne}(G_1, G_2) = \{c^* \in C_A \mid\ & u^1_i(c^* \Downarrow_{X_1}) \geq u^1_i(c \Downarrow_{X_1}) \wedge \\ & u^2_j(c^* \Downarrow_{X_2}) \geq u^2_j(c \Downarrow_{X_2}), \\ & \forall i \in X_1, \forall j \in X_2, \forall c \in C_A\}\end{aligned}$$

Let $c^* \in \mathsf{Gne}(\{G_1, G_2\})$ and $i \in A$. Because $X_1 \cap X_2 = \emptyset \wedge X_1 \cup X_2 = A$, either $i \in X_1 - X_2$ or $i \in X_2 - X_1$. Let us suppose $i \in X_1 - X_2$, then:

$$\forall c \in C_A, u_i(c) = u_i(c \Downarrow_{X_1}, c \Downarrow_{X_2}) = u^1_i(c \Downarrow_{X_1}) \leq u^1_i(c^* \Downarrow_{X_1}) = u_i(c^*))$$

Thus, $\forall i \in A, \forall c \in C_A, u_i(c) \leq u_i(c^*)$; that is $c^* \in \mathsf{pne}(G)$.
Let $c^* \in \mathsf{pne}(G)$ and $c \in C_A, i \in X_1, j \in X_2$ (thus $i \in X_1 - X_2$ and $j \in X_2 - X_1$).

$$u^1_i(c \Downarrow_{X_1}) = u_i(c \Downarrow_{X_1}, c \Downarrow_{X_2}) = u_i(c) \leq u_i(c^*) = u^1_i(c^* \Downarrow_{X_1})$$

$$u^2_j(c \Downarrow_{X_2}) = u_j(c \Downarrow_{X_1}, c \Downarrow_{X_2}) = u_j(c) \leq u_j(c^*) = u^2_j(c^* \Downarrow_{X_2})$$

So, $c^* \in \mathsf{Gne}(G_1, G_2)$, and

$$\mathsf{pne}(G) = \mathsf{Gne}(G_1, G_2)$$

Thus, the game G can be separated to $\{G_1, G_2\}$, which is false. ■

The normal form of a games network can be specified from an order on games network based on the consideration that a games network is greater than another one if the former is the result of several separations of the latter.

Definition 25 (Games Network Order).
Let $\langle \mathcal{A}_1, C_1, \mathcal{U}_1\rangle, \langle \mathcal{A}_2, C_2, \mathcal{U}_2\rangle$ be two games networks, the order $\preceq_\Omega$ is defined as follows:

$\langle \mathcal{A}_1, C_1, \mathcal{U}_1\rangle \preceq_\Omega \langle \mathcal{A}_2, C_2, \mathcal{U}_2\rangle$ if and only if :
$\mathcal{A}_1 = \mathcal{A}_2, C_1 = C_2,$
$\forall G = \langle A, u\rangle \in \mathcal{U}_1, \exists \{G_i = \langle \mathcal{A}_i, u_i\rangle\}_{i=1,n} \subseteq \mathcal{U}_2, G = \bigvee^\Omega_{i=1,n} G_i$

5.2 $\Omega-$Normal Form

We considered here Ω, the set of functions with neutral element (definition 23). The extension of the normal form to Ω will be defined according to the properties commonly shared by every functions of the class, that is, the neutral property. It is based on a new definition of the join operator as follows:

Definition 26 (Ω-Join).
Let Ω be the class of functions defined in 23. Let $\Gamma = \langle \mathcal{A}, C, \mathcal{U} \rangle$ be a games network, let $G_1 = \langle A_1, u^1 \rangle$ and $G_2 = \langle A_2, u^2 \rangle$ be two game nodes of Γ ($G_1 \in \mathcal{U}, G_2 \in \mathcal{U}$), we define:

$$G_1 \overset{\Omega}{\bigvee} G_2 = \langle A_1 \cup A_2, u \rangle$$

with :

$$\begin{aligned}
\forall c \in C_{(A_1 \cup A_2)}, \; & \forall i \in A_1 - A_2 \; u_i(c) = {u^1}_i(c \Downarrow_{A_1}) \\
& \forall i \in A_2 - A_1 \; u_i(c) = {u^2}_i(c \Downarrow_{A_2}) \\
& \forall i \in A_1 \cap A_2 \; u_i(c) = \begin{cases} {u^2}_i(c \Downarrow_{A_2})) \text{ if } {u^1}_i(c \Downarrow_{A_1}) = e_\omega \\ {u^1}_i(c \Downarrow_{A_1})) \text{ if } {u^2}_i(c \Downarrow_{A_2}) = e_\omega \\ \text{undefined} \quad \text{else} \end{cases}
\end{aligned}$$

The definition of the join operator, is now compatible with any functions of the class. Hence the separation is the same whatever the function ω is. This provides the ability to compute a function regardless to the specificity of a specific function.

Definition 27 (Games Network $\Omega-$Normal Form).
Let Γ be a games network, Ω the set of functions with neutral element.
*Γ is said to be $\Omega-$*normal *if any game node is inseparable according to the Ω join operator.*

Considering a $\Omega-$normal form obtained by separation, the following proposition shows that, for each agent, a game containing all its predecessors exists.

Proposition 10. *Let $\Gamma = \langle \mathcal{A}, C, \mathcal{U} \rangle$ be a games network, and $G = \langle A, u \rangle$ a game node of Γ. Let Ω defined in 23 and let $\Gamma_f = \langle A, C, \mathcal{U}_f \rangle$ be the $\Omega-$normal form of G obtained by separation; we have:*

$$\forall j \in A, \exists G_f = \langle A_f, u^f \rangle \in \mathcal{U}_f, \{j\} \cup \delta_u^-(j) \subseteq A_f$$

Proof. *Let Γ, G and Γ_f be defined as in lemma 10, and suppose we have $j \in A$ such that:*

$$\forall G_f = \langle A_f, u^f \rangle \in \mathcal{U}_f, \{j\} \cup \delta_u^-(j) \nsubseteq A_f$$

That is

$$\forall G_f = \langle A_f, u^f \rangle \in \mathcal{U}_f, j \notin A_f \vee (j \in A_f \wedge \exists i \in \delta_u^-(j), i \notin A_f)$$

Because Γ_f is obtained by successive separation, in one step of its construction we have a game node $G_0 = \langle A_0, u^0 \rangle$ which contains j and all its predecessors; in the next step, G_0 is separated in $G_1 = \langle A_1, u^1 \rangle$ and $G_2 = \langle A_2, u^2 \rangle$ and none of them contains j and all its predecessors.

Let us suppose j has more than two predecessors (see proposition 6 for the case where j has just one predecessor). Let i_1, i_2 be two predecessors of j: $\{i_1, i_2\} \subseteq \delta_u^-(j)$. Because of proposition 6, i_1 and j must be in a common game and i_2 and j also must be in a common game. Because not G_1 nor G_2 contains j and all its predecessors, we can suppose

$$\{i_1, j\} \subseteq A_1 \text{ and } i_2 \notin A_1$$

$$\{i_2, j\} \subseteq A_2 \text{ and } i_1 \notin A_2$$

By definition of the join operator, we have

$$\forall c \in C_{A_0}, u_j^0(c) = \begin{cases} u^2{}_j(c \Downarrow_{A_2})) \text{ if } u^1{}_j(c \Downarrow_{A_1}) = e_\omega \\ u^1{}_j(c \Downarrow_{A_1})) \text{ if } u^2{}_j(c \Downarrow_{A_2}) = e_\omega \\ \text{undefined} \quad \text{else} \end{cases}$$

And because u^0 is well defined,

$$\forall c \in C_{A_0}, u^1{}_j(c \Downarrow_{A_1}) = e_\omega \vee u^2{}_j(c \Downarrow_{A_2}) = e_\omega$$

That is either $i_2 \not\delta_u^-(j))$ or $i_1 \not\delta_u^-(j))$, which is false. ■

Many normal forms are possible given a game network. The following algorithm defines the computation of a specific normal form.

Schematically, it considers each node as a network reduced to this node and computes a well-formed normal form with it. Then, the obtained networks will be assembled to obtain a normal form of the complete network.

According to the previous proposition, each agent belongs to the same game that those on which it depends. This affects the calculation of the local payoff function. Indeed, thanks to this property (proposition 10), we can preserve the variations of the outcomes due to the predecessors in the dependence graph.

Let $G = \langle A, C, u \rangle$ be a game, let $\Gamma_f = \langle \mathcal{A}_f, C_f, \mathcal{U}_f \rangle$ be the $\Omega-$normal form of G. The payoff is defined as follows:

$$\forall G_f = \langle A_f, u^f \rangle \in \mathcal{U}_f, \forall c \in C_A,$$
$$u_j^f(c \Downarrow_{A_f}) = \begin{cases} \text{if } \delta_u^-(j) \cap A_f = \delta_u^-(j) \text{ then } u_j(c \Downarrow_{\delta_u^-(j)}, c \Downarrow_{A-\delta_u^-(j)}) \\ \text{else} \qquad\qquad\qquad\qquad e \end{cases}$$

Example 9. Let Γ be a games network, we consider the following game node

$$g = \langle \{a_1, a_2, a_3, a_4\}, u \rangle$$

where u is defined as follows:

Being given a game node $\langle A, u\rangle$, we define:

δ^- : $A \mapsto \mathbf{2}^A$ the set of predecessors in the agent dependence graph

agent : $\mathbb{N} \mapsto \mathbf{2}^A$ the set of agents connected to the game node.

pick : $C_A \times (C_A \mapsto \mathbb{R}) \mapsto \mathbb{R}$,

$\mathsf{pick}(c', u)$ gives a value $u(c)$ such that the configuration c' is contained in c.

```
function separate(⟨A, u⟩ : game node)
  U' := ∅; g := 0;
  forall i ∈ A
    g := g + 1;
    agent(g) := i ∪ δ⁻_u(i) ;
  endforall
  U = [1 : g];
  forall g' ∈ [1 : g]
    U := U − {g'' ∈ U|agent(g'') ⊂ agent(g') ∨ (agent(g') = agent(g'') ∧ g'' < g')};
  endforall
  forall g ∈ U
    forall j ∈ agent(g)
      if δ⁻_u(j) ∩ agent(g) = δ⁻_u(j) then
        forall c ∈ C_agent(g) u^g_j(c) := pick(c, u)
      else
        forall c ∈ C_agent(g) u^g_j(c) := e
      endif
    endforall
      U' = U' ∪ {⟨agent(g), u^g⟩};
  endforall
  return U';
```

Fig. 2. Normal Form Algorithm for a Game Node

a_1	a_2	a_3	a_4	u_1	u_2	u_3	u_4
.F.	.F.	.F.	.F.	0	0	1	1
.F.	.F.	.F.	.T.	0	0	1	0
.F.	.F.	.T.	.F.	0	0	0	1
.F.	.F.	.T.	.T.	0	0	0	0
.F.	.T.	.F.	.F.	1	2	1	0
.F.	.T.	.F.	.T.	1	2	1	1
.F.	.T.	.T.	.F.	1	2	0	0
.F.	.T.	.T.	.T.	1	2	0	1
.T.	.F.	.F.	.F.	2	1	0	1
.T.	.F.	.F.	.T.	2	1	0	0
.T.	.F.	.T.	.F.	2	1	1	1
.T.	.F.	.T.	.T.	2	1	1	0
.T.	.T.	.F.	.F.	0	0	0	0
.T.	.T.	.F.	.T.	0	0	0	1
.T.	.T.	.T.	.F.	0	0	1	0
.T.	.T.	.T.	.T.	0	0	1	1

From the table describing u, we can deduce the following dependencies

$$a_1 \delta a_3, a_2 \delta a_4, a_1 \delta a_2, a_2 \delta a_1$$

According to the algorithm and from the dependence graph, we can deduce that the game node is separated into three games nodes, each one having 2 agents. Figure 3 describes the resulting games network. Each game node is denoted by $g_{i,j} = \langle \{i, j\}, u \rangle$.

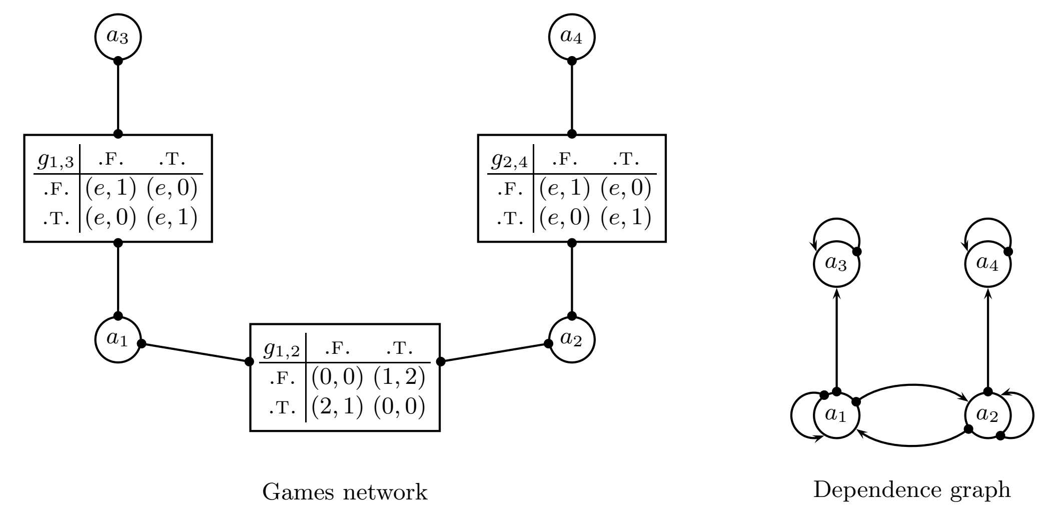


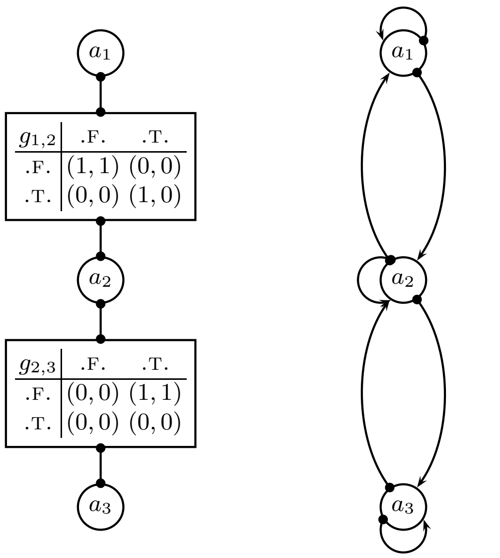
Fig. 3. Normal form of the games network of example 9

6 Discussion

The inherent constraints in the equilibria and the reorganization of the games networks underline phenomena which cannot be emphasized by the game theory. They show the potential limitation of the observations due to the process of the coupling.

For instance, if we consider the following game (figure 4), we first see that it cannot be the consequence of a separation of a strategic game (because of the dependences). However it may be joined while preserving equilibria, if we consider a chosen instance of the join (max for the example). In this case, we remark that we cannot distinguish the role of a_1 from that of a_3 when they interact with a_2 because both interactions are merged (mathematically by $\bigvee^{\max}$ operation). In the games network, some combinations of strategies are not allowed. For example, if $a_1 = \texttt{.F.}$ and $a_2 = \texttt{.F.}$ we cannot have $a_3 = \texttt{.F.}$ because $u_2^{g_{1,2}}(\texttt{.F.}, \texttt{.F.}) = 1$ and $u_2^{g_{2,3}}(\texttt{.F.}, \texttt{.F.}) = 0$, which is incompatible. But, in the merged game, all the combinations are allowed, resulting in differences on some payoff functions. Thus, in the games network $u_2 = 1$ iff $\neg a_2 \wedge (\neg a_1 \wedge a_3)$ is true; whereas in the join game $u_2 = 1$ iff $\neg a_2 \wedge (\neg a_1 \vee a_3)$ is true.

Relating to system modeling, the games networks show how degrees of freedom of a system, symbolized by the Nash equilibria, are reduced by the coupling between modules. It emphasizes the problem involved in analysis of an individual sub-system when observed phenomena disappear once they are plunged into the whole system. In the genetic network analysis, the term "nonfunctional" describes this kind of situations [19]. The theory of the games networks proposes an explanation to this non-functionality by the reduction of equilibria due to the coupling of the games.



Games network *Dependence graph*

a_1	a_2	a_3	u_1	u_2	u_3
.F.	.F.	.F.	1	1	0
.F.	.F.	.T.	1	1	1
.F.	.T.	.F.	0	0	0
.F.	.T.	.T.	0	0	0
.T.	.F.	.F.	0	0	0
.T.	.F.	.T.	0	1	1
.T.	.T.	.F.	1	0	0
.T.	.T.	.T.	1	0	0

$g_{1,2} \bigvee^{\max} g_{2,3}$

Fig. 4. Example

7 Conclusion

In this article, we proposed an extension of the game theory, named theory of the games networks, which provides a framework to model modular interactions. It accounts for the phenomenon of coordination between agents in front of localized modular interactions.

This phenomenon and its consequences are not emphasized by the games theory because the payoff function is determined by considering all the strategies. Hence, the concept of modular interaction is not explicitly represented.

Concerning the calculation of equilibrium, this extension puts forward two categories of constraints: functional constraints, based on the games nodes, which underline the Nash equilibria; and structural constraints, based on the interconnection of the agents to different games and on the single play rule in strategic games, which results in finding strategies compatible with the equilibria of a given set of games nodes.

Within this framework, we endeavored to define the conditions which make it possible to establish structural equivalences between the games networks, equivalences based on the conservation of equilibria. These conditions show the importance of the observer (represented by the function ω). To a certain extent, the reorganization is comparable from the standpoint of an observer on the modelled system. To gain in independence with respect to an observer, we proposed a method which does not consider a particular function, but a class of functions. From it, we deduced an algorithm to reorganize networks. The automatic reorganization as much as possible subdivides the nodes of games until obtaining a normal form. This process is used to indicate the details of the structure leading to the same equilibria as the initial games network (or part of this one). We have applied games network theory to analyse the structure of biological networks which carry the regulatory biological process [4, 6].

We have based the theory of games networks on the strategic games which are the core of various variations of the games theory (sequential, repeated, bayesian, evolutionary). This contributes to present the main aspects of dynamics relating to the equilibria – that is, the formation of the total equilibria from local equilibria and the conditions of invariant reorganizations in equilibria. A perpective for this work is to consider another kind of models to describe games nodes. The goal is to investigate the impact of the modularity on other aspects of the dynamics covered by the games theories.

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