On the Relationships between Interactions and Equilibria in Games Networks

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Abstract

In this paper we present an extension of game theory named *games network*. The objective of this extension is to propose a theoretical framework which suits to "modular dynamics" resulting from different local interactions between agents. Briefly, games networks describe the situation where each player plays different games at the same time with several different players. The theoretical extension is based on strategic games. We more particularly focus on the determination of a global equilibrium from local Nash equilibria. Especially we determine some conditions on the existence of a global mixed equilibrium whatever could be the representation of given interactions by a game network. The results are based on an intermediary representation called the *dependence graph* which describes interactions between players.

Key words: Complex networks dynamics, Games theory, modularity.

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1 Introduction

Game theory [1] provides a modeling framework to study complex interplays between agents (or players). It was used in a large variety of fields such as Biology [2,3], Economy [4,5], Computer Science [6,7] to analyse complex interactions. *Strategic games* describe interactions between agents by associating a payoff to all possible played configurations for each agent. The design of a game is based on the assumption that every agents interact with each other.

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Hence, it does not consider the locality of interactions which often occurs in real systems. The analysis of the locality emphasizes the study of groups of highly interacting agents. Such groups are called *modules* and appear to play a significant rule in the analysis of systems as in biology for instance [8]. Analyzing activities of a system in light of modularity appears to be a central step on the understanding of a system because it emphasizes the organization of a system from its sub systems.

To tackle with the description of the interaction-based modularity, we propose an original theoretical framework named *games network* [3]. The games network theory extends game theory by considering that agents can be involved in different games at the same time. Consequently, each game to which an agent participates must be considered to determine its strategy. Informally, a games network is represented by a bipartite graph where an agent is connected to each game (node) it plays (cf. figure 1). The games networks will be used to analyze local interacting processes between agents. It aims at determining global steady states of an interaction based system by considering each local interplays. The definition of games networks enables us to structure interactions between agents into modules (game nodes).

The computation of an equilibrim in a game relies of the computation of a Nash equilibrium [9] which is a central result of game theory. In modeling, it is often assimilated to a steady state ([10]). From local Nash equilibria, one may define a global equilibrium. Schematically, a Nash equilibrium exists whatever the game but a global equilibrium related to a specific network may not exist, because the decision of some agents can be contradictory for two games for instance. In this paper we propose to define some relationships between the interactions of agents and the existence of global equilibria.

The paper is organised as follows: Section 2 deals with related work. Section 3 presents notations used in the article. Section 4 briefly recalls the main results on strategic game theory. Section 5 presents the extension of strategic games to games networks and deals with the combination of games in a games network. We define the notion of *mixed games network equilibrium* which corresponds to a global equilibrium in a games network. Section 6 interested in the relation between organization and global equilibria.

2 Related work

Research of the steady states of a game, and so computation of Nash equilibria, is certainly one of the most studied field in game theory. McKelvey and

[3] English translation of *réseau de jeux*

McLennan note that the computation of Nash equilibria in n-players games is much harder, in many important ways, that the computation in 2-players games.

Recently, some authors were interested in a new way to reduce the complexity of Nash equilibria computation. A game is not considered in its globality, but through the local interactions between the players of the game. La Mura, in [11], to treat multi-agent decision problems, introduces a new game representation, more strutured and more compact than classical representations in game theory (normal or extensive form for example). Considering the strategic separabilities in its representation, La Mura presents convergence methods to compute Nash equilibria. Interested in Bayesian networks, Koller and Milch in [12] propose a representation language for general multi-player games named Multi-Agent Influence Diagrams. They insist on the importance of dependence relationship among variables to detect structures in games and decrease the computational cost of finding Nash equilibria. Kearns, Littman and Singh in [13] introduce a compact graph-theoretic representation for multi-party game theory. Their main result is an efficient algorithm for computing approximate Nash equilibria in one-stage games represented by trees or sparse graphs.

In this paper, we focus on interactions localised to a given process. Our games network representation, compared to La Mura, is not another game-theoritic representation but an extension of strategic representation. The closest representation is that of Kearns, Littman and Singh. However, in quite a some way as Koller and Milch, we are interested in the influence of the network organization on the existence of Nash equilibria on the scale of the network.

3 Notations

In the paper, we use the following notations:

- $[a : b] = \{i \in \mathbb{Z} | a \leq i \leq b\}$ denotes a discrete interval bounded by a and b.
- $|A|$ denotes the cardinal of a set A.
- Let $i \in A$, i also denotes the singleton $\{i\}$ if it is required by the context of the operation.
- Let $A = [1 : n]$, given $C = \{C_i\}_{i \in A}$, we note:
  - · $C_A = \times_{i \in A} C_i$
  - · $C_X = \times_{i \in X} C_i, \forall X \subseteq A$
  - · $C_{-j} = \times_{i \in A-j} C_i, \forall j \in A$
- We consider the lifted version $C_{\mathbf{lift}} = C + \{\perp\}$ where the element *Bottom* denoted by $\perp$ is added to C.
- Concerning the profiles or vectors, we adopt the following notations. Given $A = [1 : n]$, given a profile $c \in C_A$ of a set $C_A = \times_{i \in A} C_i$, we denote by:

- $c_{-i} = (c_1, \cdots, c_{i-1}, c_{i+1}, \cdots, c_n)$; this excludes the i^{th} component of a profile.
- $(c_{-i}, \mathbf{c}_i) = (c_1, \cdots, c_{i-1}, \mathbf{c}_i, c_{i+1}, \cdots, c_n)$. The notation distinguishes the i^{th} component of the profile from the others. This notation is extended to sets of indices, $(c_{-X}, c_X), X \subset [1 : n]$.

4 Strategic Games

In this section we give definitions of the game theory used in the article. The reader may refer to the books [14–16] for a complete overview of the game theory and its applications.

4.1 Definition of a strategic game

Strategic game is a model of interplays where each agent chooses its plan of action (or strategy) once and for all, and these choices are made simultaneously. Moreover, each agent is rational and perfectly informed of the payoff function of other agents. Thus, they aim at maximizing their payoffs while knowing the expectation of other agents.

Definition 4.1 (Normal or Strategic Representation)
A strategic game Γ is a $3-$uple $\langle A, C, u\rangle$ where:

- *A is a set of players or agents.*
- *$C = \{C_i\}_{i\in A}$ is a set of strategy sets where each C_i is a set of strategies available for the agent i, $C_i = \{c_i^1, \cdots, c_i^{m_i}\}$.*
- *$u = (u_i), i \in A$ is a vector of functions where each $u_i : C \mapsto \mathbb{R}, i \in A$ is the payoff function of the agent i.*

In order to conveniently combine sets of strategies, we define the strategy as follows:

Definition 4.2 (Set of Strategies)
Let $\langle A, C, u\rangle$ be a strategic game, let Φ^* be a set of labels, The set of strategies $C = \{C_i\}_{i\in A}$ are defined as follows $\forall i \in A, C_i = \{(i, \varphi) | \varphi \in \Phi^*\}$.

By this definition, the fact that agents share the same strategies do not interfere in the union of sets of strategies.

4.2 Mixed (or Randomized) strategies

Given a strategic game $\Gamma = \langle A, C, u\rangle$, a *mixed-strategy*[4] for any player i is a probability distribution over C_i. We let $\Delta(C_i)$ denote the set of all possible mixed strategies for player i.

$$\Delta(C_i) = \{(p_j)_{j\in[1:m_i]} | \forall j \in [1 : m_i], 0 \leq p_j \leq 1 \wedge \sum_{j=1}^{m_i} p_j = 1\}$$

A *mixed-strategy profile*[5] σ is any vector that specifies one mixed strategy $\sigma_i \in \Delta(C_i)$ for each agent $i \in A$. We let $\Delta(C)$ denotes the set of all possible mixed-strategy profiles.

$$\Delta(C) = \times_{i\in A}\Delta(C_i)$$

For any mixed-strategy profile $\sigma \in \Delta(C)$, let $u_i(\sigma)$ denotes the payoff for player i.

$$u_i(\sigma) = \sum_{c\in C}(\prod_{j\in A} \sigma_j(c_j))u_i(c), \forall i \in A$$

4.3 Nash Equilibrium

Nash equilibrium is the central concept of the game theory. This notion captures the steady states of the play of a strategic game in which each agent holds the rational expectation about the other players behavior. A *mixed Nash equilibrium* is defined as follows:

Definition 4.3 ((Mixed) Nash Equilibrium of a Strategic Game)
Let $\langle A, C, u\rangle$ be a strategic game, a mixed Nash equilibrium[6] *is a mixed-strategy profile σ^* with the property that :*

$$\forall i \in A, \forall \sigma_i \in \Delta(C_i), u_i(\sigma^*_{-i}, \sigma_i) \leq u_i(\sigma^*_{-i}, \sigma^*_i)$$

In other words, *no agent can unilaterally deviate of a mixed Nash equilibrium without decreasing its payoff.*

[4] If the distribution is such that only one probability is different to 0, then the mixed-strategy is called *pure strategy.*
[5] If the strategy of each player is pure, the profile is said to be pure.
[6] If the profile is pure, we speak about *pure Nash equilibrium.*

Definition 4.4 (Set of Mixed Nash Equilibria)
Let $G = \langle A, C, u \rangle$ be a game, we define $\mathbf{mne}(G)$*, the set of mixed Nash equilibria for G:*

$$\mathbf{mne}(G) = \{\sigma^* \in \Delta(C) | u_i(\sigma^*_{-i}, \sigma_i) \leq u_i(\sigma^*_{-i}, \sigma^*_i), \forall i \in A, \forall \sigma_i \in \Delta(C_i)\}$$

5 Games Network

A games network corresponds to an extension of the game theory which defines *modular interactions* localized to different subsets of agents. Each module corresponds to a specific game defined by a payoff function. Parameters of the payoff function are strategies of agents involved in the game. Agents are shared between different modules and played different games in parallel. However, they have the same set of strategies for every games they played. In a games network, several games are combined to form a more general structure of network. In this section, we address the main definitions of a games network. The reader may refer to [17] for a more complete overview.

5.1 Definition of a Games Network

The definition of a games network mainly consists of defining a set of agents connected to a set of games. The normal form of a games network is as follows:

Definition 5.1 (Games Network)
A games network is a $3-$*uple* $\langle \mathcal{A}, C, \mathcal{U} \rangle$ *where*

- $\mathcal{A}$ *is a set of agents or players.*
- $C = \{C_i\}_{i \in \mathcal{A}}$ *is a set of sets of strategies.*
- $\mathcal{U} = \{\langle A, u \rangle\}$ *is a set of* game nodes *where each* $A \subseteq \mathcal{A}$ *is a set of agents and* $u : A \times C_A \mapsto \mathbb{R}$ *is a set of payoff functions such that* $u = \{u_i : C_A \mapsto \mathbb{R}\}_{i \in A}$.

A games network offers a synthetic representation to define the different interplays between several players. The structure $\langle A, u \rangle$ totally determines a game played by a subset of agents since it useless to include the strategies which are the same for any agent of the network. A games network is represented by a bipartite graph $\langle \mathcal{A}, \mathcal{U}, E \rangle, E \subseteq \mathcal{A} \times \mathcal{U}$ where an edge $(i, \langle A, u \rangle)$ is a member of E if and only if $i \in A$ (See figure 1 for an illustration).

5.2 Restriction

A game node can be viewed as a sub game of a larger game played by the whole agents of the network. To focus on an arbitrary sub game, we equip the theory with the *restriction operator*

Definition 5.2 (Mixed-strategy Profile Restriction)
Let $\mathcal{A} = [1 : n]$ be a discrete interval representing a set of agents, let $C = \{C_i\}_{i\in\mathcal{A}}$ be a set of strategy sets. Given a mixed-strategy profile $\sigma \in \Delta(C)$ [7]*, we define its restriction to a subset $A \subseteq \mathcal{A}$, denoted by $\sigma{\downarrow_A}$: $\Delta(C) \times \mathbf{2}^A \mapsto \Delta(C)_{\mathbf{lift}}$, as follows* [8] *:*

$$(\sigma{\downarrow_A})_i = \begin{cases} \sigma_i & \text{if } i \in A \\ \perp & \text{otherwise} \end{cases}$$

We extend the restriction operator by removing bottom elements ($\perp$) of the profile, but the order of the other values is conserved in the resulting profile. We note the composition of the removals and restriction operation as follows: $\sigma \Downarrow_X$

Example 5.1 *Let $\mathcal{A} = [1 : 4]$ and $\sigma = (\sigma_1, \sigma_2, \sigma_3, \sigma_4)$. Let $A = \{1, 3\}$, we have $\sigma{\downarrow_A} = (\sigma_1, \perp, \sigma_3, \perp)$ and $\sigma \Downarrow_A = (\sigma_1, \sigma_3)$.*

The restriction is obviously extended to a set of mixed-strategy profiles by applying the operation to every elements.

The previous definition (5.2) restricts the mixed-strategy profile to relevant values according to a subset of agents, named its *support*. A profile of values defined by a restriction is considered as a *local profile* of a subset of agents. Whatever the values associated to other agents are, they will not be considered for a local profile.

The restriction applied to mixed-strategy profiles will be used in the next section to put the focus on a sub part of a profile which corresponds to a games node.

5.3 Mixed Games Network Equilibrium

The definition of a games network allows the combination of several games into a single network. This puts the emphasis on the way that the network struc-

[7] Recall that $\Delta(C)$ denote the set of all possible mixed-strategy profiles
[8] $\perp$ stands for an irrelevant value

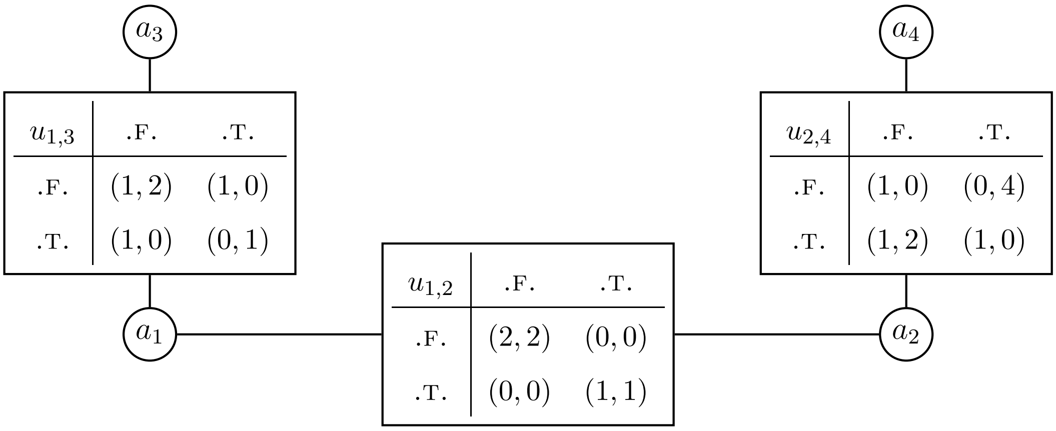

Fig. 1. The games network from section 5.4

ture is determined, because different structures can be proposed to model the same situation. In order to compare them, it is necessary to identify the equivalence between games networks. The conditions of equivalence investigated in the paper are based on the equilibria. Informally, two games are equivalent if their equilibria are the same. Such a condition requires to enlarge the equilibrium locally computed from game nodes to the whole games network. The equilibrium at the scale of the network is named the *mixed games network equilibria* (**MGne**).

A games network equilibrium corresponds to a compatible association of local equilibria. We assume that agents follow the *single played strategy* rule, that is an agent plays the same strategy for every connected games. The definition of **MGne** can of course be applied to the whole network, but the restriction to a subset of game nodes allow us to define regions where equilibria are compatible.

Definition 5.3 (Mixed Games Network Equilibrium)
Let $\Gamma = \langle \mathcal{A}, C, \mathcal{U} \rangle$ be a games network, let $c^* = (c_1, \cdots, c_n)$ be a strategy profile of every agents[9] *. c^* is a mixed games network equilibrium of a subset $U \subseteq \mathcal{U}$ (noted $c^* \in \mathbf{MGne}_\Gamma(U)$) iff:*

$$\forall \langle A, u \rangle \in U, c^* \Downarrow_A \text{ is a mixed Nash equilibrium of the game } \langle A, (C_i)_{i \in A}, u \rangle$$

5.4 An example of games network

Let us consider $\Gamma = \langle \mathcal{A}, C, \mathcal{U} \rangle$ the games network of figure 1. We have:

- $\mathcal{A} = \{a_1, a_2, a_3, a_4\}$, the agents

[9] Recall that by convention $|\mathcal{A}| = n$.

- $C_i = \{.\text{F}., .\text{T}.\}, \forall i \in \mathcal{A}$, the strategies of the agents
- $\mathcal{U} = \{\langle A_{1,3}, u^{1,3}\rangle, \langle A_{1,2}, u^{1,2}\rangle, \langle A_{2,4}, u^{2,4}\rangle\}$, the game nodes where $A_{1,3} = \{a_1, a_3\}$, $A_{1,2} = \{a_1, a_2\}$, $A_{2,4} = \{a_2, a_4\}$ and the payoffs functions are shown in figure 1.

To compute the **MGne** of Γ, let us compute the **mne** of each sub-game.
$\mathbf{mne}_{1,3} = \mathbf{mne}(\langle A_{1,3}, u^{1,3}\rangle) = \Big\{\Big((1,0),(1,0)\Big)\ ;\ \Big((\frac{1}{3},\frac{2}{3}),(1,0)\Big)\Big\}$

$\mathbf{mne}_{1,2} = \mathbf{mne}(\langle A_{1,2}, u^{1,2}\rangle) = \Big\{\Big((1,0),(1,0)\Big)\ ;\ \Big((0,1),(0,1)\Big)\ ;\ \Big((\frac{1}{3},\frac{2}{3}),(\frac{1}{3},\frac{2}{3})\Big)\Big\}$

$\mathbf{mne}_{2,4} = \mathbf{mne}(\langle A_{2,4}, u^{2,4}\rangle) = \Big\{\Big((0,1),(1,0)\Big)\ ;\ \Big((\frac{1}{3},\frac{2}{3}),(1,0)\Big)\Big\}$

Thus, we can compute the **MGne** of Γ:

$$\mathbf{MGne}_\Gamma = \Big\{\Big((\frac{1}{3},\frac{2}{3}),(\frac{1}{3},\frac{2}{3}),(1,0),(1,0)\Big)\Big\}$$

6 Relations between interactions and MGne

The general existence theorem of Nash ([9]) indicates us that, given any finite game, there exists at least one mixed Nash equilibrium. On the contrary, considering any games network, a mixed games network equilibrium may not exist because the decision of some agents could be contradictory for 2 games.

In the section, we investigate the relationships between *interactions* and global equilibria. Interactions correspond to relationships between agents which are computed from game nodes, they are related to the payoff function. Informally, we express the interaction as a *dependence relation*: an agent depends on another if modification of the strategies of the latter induces a variation of the payoffs of the former. In real systems, such as biological systems, we are able to define the interactions. However, there is no one-to-one mapping between dependence graph and games network. In some extend the games network can be viewed as a specific organization of interactions. Hence, it appears interesting to put the emphasis on the relationships between equilibria and dependence graph, because it helps us to determine whether an equilibrium specifically depends on a games network or only depends on the structure of interactions.

6.1 Dependence

To precisely describe the interplays occurring in a game, we define the notion of *dependence* between agents. Informally, an agent is dependent on another

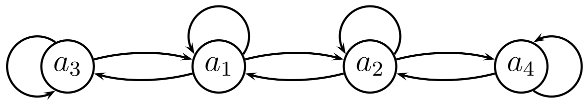

Fig. 2. Dependence graph for the games network from figure 1

if its payoffs are altered by the strategies of the other player.

Definition 6.1 (Agent dependence)
Let $\langle A, C, u\rangle$ be a strategic game, let $j, i \in A^2, i \neq j$ be two agents. j is said to be dependent *on i, denoted by $i\delta_u j$, if:*

$$\exists c_i \in C_i, \exists c'_i \in C_i, \exists c_{-i} \in C_{-i}, u_j(c_{-i}, c_i) \neq u_j(c_{-i}, c'_i)$$

The dependences provide an overview of the interplays of the agents in a game without having carefully studying the payoff function. To get an abstraction of the dependences according to a game, we introduce a new representation named *the agent dependence graph.*

Definition 6.2 (Agent Dependence Graph)
Let $G = \langle A, C, u\rangle$ be a strategic game, the agent dependence graph $D_G = \langle A, E\rangle$ is a graph such that: $E = \{(i, j)|i\delta_u j\}$

The dependence relation for a game is extended to the dependence relation by considering a games network as follows:

Definition 6.3 (Dependence relation according to a games network)
Let $\Gamma = \langle \mathcal{A}, C, \mathcal{U}\rangle$ be a games network, let $i \in \mathcal{A}$ and $j \in \mathcal{A}$ be two agents,

$$i\delta_{\mathcal{U}} j \text{ iff } \exists G = \langle A, u\rangle \in \mathcal{U} \text{ such that } i\delta_u j$$

(Definition of dependence graph is extended in the same way.)

Figure 2 shows the dependence graph for the games network from figure 1.

6.2 Dependence and **MGne**

We more precisely analyse the interactions between dependence and existence of mixed games networks equilibria. Obviously, several games networks can have the same dependence graph, but theorem 6.1 gives a result, considering a particular dependence graph, and whatever could be the games network having such a dependence graph. Thus, theorem 6.1 is interested in the dynamics of the games network and particulary in the *lack of self-dependent agent* (that is, if we consider a games network $\Gamma = \langle \mathcal{A}, C, \mathcal{U} \rangle$, the lack of agents in $\{i \in \mathcal{A}, i\delta_{\mathcal{U}} i\}$).

Theorem 6.1 (Simple dependence graph)
Let $\Gamma = \langle \mathcal{A}, C, \mathcal{U} \rangle$ be a games network and D_Γ be its dependence graph. If D_Γ is simple, that is if there is no self-loop in D_Γ, then Γ has an infinite number of mixed games network equilibria and

$$\mathbf{MGne}_\Gamma(\mathcal{U}) = \Delta(C)$$

Proof. See proof 1 in Annexe.

Lemma 6.1 and theorem 6.2 are interested in *non-self-dependent agents* (that is agents in $\{i \in \mathcal{A} | i \not\delta_{\mathcal{U}} i\}$). Lemma 6.1 allows us to determine that non-self-dependent agents does not make it possible the decrease of the number of Nash equilibria. Theorem 6.2 extends this result to games networks and mixed conditions on structure and on organization to provide a sufficient condition to existence of a mixed games network equilibrium.

Lemma 6.1 (Non-self-dependent agents)
Let $G = \langle A, C, u \rangle$ be a game. Let $N \subseteq A$ the set of non-self-dependent agents, $N = \{i \in A | i \not\delta_u i\}$.

$$\forall c_N \in C_N, \exists c_{-N} \in C_{-N}, (c_{-N}, c_N) \in \mathbf{mne}(G)$$

Proof. See proof 2 in Annexe.

Theorem 6.2
Let $\Gamma = \langle \mathcal{A}, C, \mathcal{U} \rangle$ be a games network. If agents participating to several games are non-self-dependent, then it exists at least one mixed games network equilibrium.

Proof. The proof is an immediate consequence of Lemma 6.1.

Conclusion

In this article we propose an extension of the game theory, named theory of the games networks, which provides a framework to model modular interactions. Within this framework, we endeavored to define the conditions which make it possible to establish structural equivalences between games networks, equivalences based on the conservation of global equilibria. We finally interested in conditions based on the interactions between agents to determine whether or not a global equilibrium may exist.

A global equilibrium corresponds to a combination of local equilibria. In other words, we can construct a games network equilibrium if we compute a local equilibrium of a game and if we "forward" this equilibrium to the other games connected to the previous one by a common agent (the "transmission" have to be respectful of the single played strategy rule).

We put the focus on two kind of agents, self-dependent agents and non-self-dependent ones. Whereas non-self-dependent agents allows the transmission of all the strategies, whatever they are, self-dependent agents filter compatible strategies. That is self-dependent agents only allow global equilibria compatible with the local equilibria of the game the self-dependent agents are connected to. We can approach this situation of the phenomenon of transmission of the signal in modeling. Thus, we can see self-dependent agents as creators of a signal and non-self-dependent agent as transmitters. A global equilibrium exists only if the different signals, created by the self-dependent agents are compatible between them.

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Annexe

Proof 1 (Theorem 6.1)
By definition we have $\mathbf{mne}_\Gamma(\mathcal{U}) \subseteq \Delta(C)$*. To show the other inclusion, let us consider a strategic game whose dependence graph is simple, and let us show that any mixed strategy is a Nash equilibrium of this game. Thus, by composition of Nash equilibria, we can deduce that any mixed strategy of* Γ *is a network equilibrium.*

Let $G = \langle A, C, u\rangle$ be a strategic game and let D_G its dependence graph which is simple. Because the dependence graph is simple, we have $\forall i \in A, i \not\partial i$, that is

$$\forall i \in A, \forall c_i, c'_i \in C_i, \forall c_{-i} \in C_{-i}, u_i(c_{-i}, c_i) = u_i(c_{-i}, c'_i)$$

Thus, given $\sigma^* \in \Delta(C)$ and $i \in A$, we have

$$\begin{aligned} u_i(\sigma^*) &= \sum_{c \in C}\Big(\big(\prod_{j\in A}\sigma^*_j(c_j)\big).u_i(c)\Big) \\ &= \sum_{c_{-i}\in C_{-i}}\sum_{c_i\in C_i}\Big(\big(\prod_{j\in A-i}\sigma^*_j(c_j)\big).\sigma^*_i(c_i).u_i(c_{-i},c_i)\Big) \\ &= \sum_{c_{-i}\in C_{-i}}\Big(\big(\prod_{j\in A-i}\sigma^*_j(c_j)\big)\sum_{c_i\in C_i}\big(\sigma^*_i(c_i).u_i(c_{-i},c_i)\big)\Big) \end{aligned} \tag{1}$$

Because $u_i(c_{-i}, c_i) = u_i(c_{-i}, c'_i)\forall c_i, c'_i \in C_i$, and $\sum_{c_i\in C_i}\sigma^*_i(c_i) = 1$, we have

$$u_i(\sigma^*) = \sum_{c_{-i}\in C_{-i}}\Big(\big(\prod_{j\in A-i}\sigma^*_j(c_j)\big).u_i(c)\Big)$$

which does not depend on σ_i. Thus

$$\forall\sigma^* \in \Delta(C), \forall i \in A, \forall\sigma_i \in \Delta(C_i), u_i(\sigma^*_{-i}, \sigma_i) \leq u_i(\sigma^*_{-i}, \sigma^*_i)$$

That is σ^* is a Nash equilibrium for G. ■

Proof 2 (Lemma 6.1)
Let $c^*_N \in C_N$ be a strategic profile for the non-self-dependent agents. We are going to construct $c^* \in C$ such that $c^* \in \mathbf{mne}(G)$. Let us consider the game $G_{-N} = \langle A_{-N}, C_{-N}, u_{-N}\rangle$ with

- $A_{-N} = A \setminus N$,
- $C_{-N} = \times_{i\in A\setminus N}C_i$,
- $u_{-N} : C_{-N} \mapsto \mathbb{R}, u_{-N}(c_{-N}) = u(c_{-N}, c^*_N)$.

We know it exists at least one nash equilibrium for game G_1. Let $c^*_{-N} \in$ $\mathbf{mne}(G_{-N})$ such an equilibrium and let $c^* = (c^*_{-N}, c^*_N)$. Then,

$$\forall i \in N, \forall c_i \in C_i, u(c^*_{-i}, c_i) = u(c^*_{-i}, c^*_i) \text{ because } i \not\partial i$$

and

$$\forall i \in A \setminus N, \forall c_i \in C_i, u(c^*_{-i}, c_i) = u_{-N}(c^*_{-N-i}, c_i) \leq u_{-N}(c^*_{-N-i}, c^*_i) = u(c^*_{-i}, c^*_i)$$

That is $\forall i \in A, \forall c_i \in C_i, u(c^*_{-i}, c_i) \leq u(c^*_{-i}, c^*_i)$ and $c^* \in \mathbf{mne}(G)$ ■